SYMMETRIC SPACES, TOPOLOGY, AND LINEAR ALGEBRA

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Abstract. We will discuss three subjects:
1. Geometry of symmetric spaces
2. Relations to algebraic topology
3. Links to linear algebra

The linear algebra happens over the real division algebras, the real, complex, quaternionic and octonionic numbers. The relation of symmetric spaces to octonions is least understood since a "linear algebra" over the octonions does not (yet) exist. This is ongoing work.

1. Introduction

Riemannian geometry is locally just nearly-euclidean geometry with all its features. The local quantity distinguishing Riemannian from euclidean geometry is the curvature tensor R. Thus the most basic Riemannian spaces are those where R is "constant", that means parallel. These are (at least locally) the Riemannian symmetric spaces. There are other geometric descriptions of these beautiful objects: They allow an isometric point reflection s_p (called "symmetry") at any point p, and parallel transport happens by isometries, like in euclidean space (translations). Algebrically they can be described by pairs (G, σ) where G is a Lie group and σ an order-2 automorphism (involution) of G with compact fixed group $K \subset G$. Riemannian symmetric spaces are all known; they have been classified 90 years ago by Elie Cartan.

What makes these spaces interesting not only for specialists in Riemannian geometry is their close connection to other fundamental mathematical theories, in particular topology and linear algebra. On the one hand, symmetric spaces are related to a very general topological theorem, Bott's periodicity theorem, a fundamental statement for homotopy theory of classical groups and for vector bundles and K-theory; these belong to the most common tools in algebraic topology.

On the other hand, there is a twofold link to linear algebra. (1) Symmetric spaces are related to normal form problems in linear algebra, e.g. the diagonalization of self adjoint matrices or the singular value decomposition of arbitrary real matrices. (2) The "classical" examples form important structures for the linear algebra over real, complex or quaternionic numbers: they are sets of subspaces (Grassmannians) or substructures. However, there are also some spaces which do not fit into this scheme. They seem to be related to the remaining division algebra (beneath the real, complex, quaternionic numbers), the octonions, but the relation is not so clear since there yet exists no linear algebra over the octonions.

All three subjects: Geometry, Topology, Linear Algebra, shall be addressed in the talks.

2. Symmetric Spaces

The fundamental object of Riemannian geometry is the Riemannian metric on a smooth manifold P, an inner product $g = \langle\, , \,\rangle$ on any tangent space T_pP, depending smoothly on p. This allows to define angles, path length and distance (the length of the shortest path). It even defines parallel transport of tangent vectors along any path. and hence a differentiation ∇ of tangent vector fields along paths. This was

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introduced hundred years ago by Tullio Levi-Civita and is called the Levi-Civita (LC) derivative.

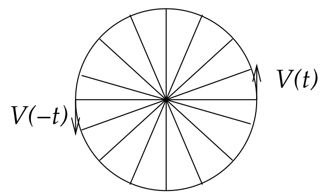
Locally, a manifold is just an open subset $U \subset \mathbb{R}^n$, and on this set, g is a positive definite symmetric matrix $g_{ij} = g(e_i, e_j)$ depending smoothly on $u \in U$. A special case is the euclidean metric $g_{ij} = \delta_{ij}$. If we change coordinates, g is expressed by a different matrix, and we can not easily recognize the euclidean metric in curved coordinates. But the LC derivative gives us a quantity which distiguishes between Riemannian and euclidean geometry in any coordinate system. Let ∇_i be the LC derivative along the i-th coordinate line, the i-th partial LC-derivative. Recall that usual partial derivatives commute, $\partial_i\partial_j = \partial_j\partial_i$. Not so for LC derivatives: The commutator $R_{ij} = [\nabla_i, \nabla_j] = \nabla_i\nabla_j - \nabla_j\nabla_i$ is nonzero in general, however it is an algebraic object, a linear map on each tangent space, which does not involve any differentiation. This is called the *Riemannian curvature tensor*; it vanishes (in any coordinate system) if and only if g is euclidean.

One of the geometric meanings of the curvature tensor is the separation of locally shortest paths (geodesics). Like a line in euclidean space, a geodesic is a path γ whose tangent vector γ' is parallel along γ. If $(\gamma_s)_{s\in(-\epsilon,\epsilon)}$ is a smooth one-parameter variation of a geodesic $\gamma = \gamma_0$ by other geodesics and $V(t) = \frac{\partial}{\partial s}\gamma_s(t)|_{s=0}$ its variation vector field along γ, then

$$V'' + R(V, \gamma')\gamma' = 0 \tag{1}$$

up to parallel transport along γ, where $R(v, w) = \sum_{ij} v_i w_j R_{ij}$ for any tangent vectors v, w with coordinates v_i, w_j.

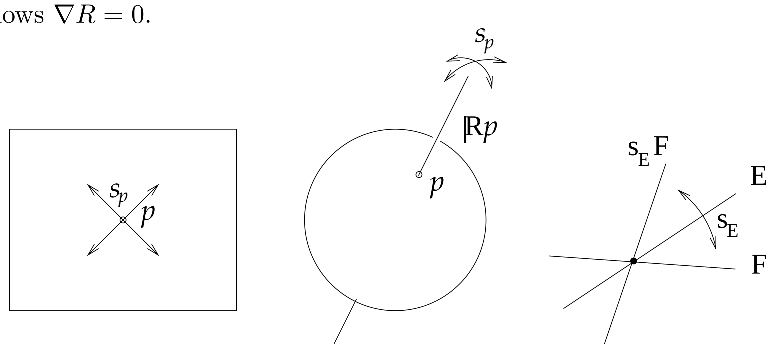
We have seen that euclidean space is characterized by $R = 0$. The most fundamental non-euclidean Riemannian manifolds should be those where R is constant, which means $\nabla R = 0$. These are the (locally) *symmetric spaces*. Then the linear ODE (1) has constant coefficients and can be solved explicitly. In particular, if $V(0) = 0$ we have $V(-t) = -V(t)$; in particular $V(t)$ and $V(-t)$ have the same length. Thus the geodesic reflection s_p (mapping $\gamma(t)$ to $\gamma(-t)$ for all geodesics γ through $p = \gamma(0)$) is an isometry, at least locally near p. It is also called *symmetry* at p.



Vice versa, if P is a Riemannian manifold such that the point reflection s_p (reflecting every geodesic through p) is an isometry for any $p \in P$, then $\nabla R = 0$. In fact, ∇R is invariant under every isometry of P, in particular under s_p. But s_p reflects each tangent vector $a \in T_pM$ into $-a$, hence

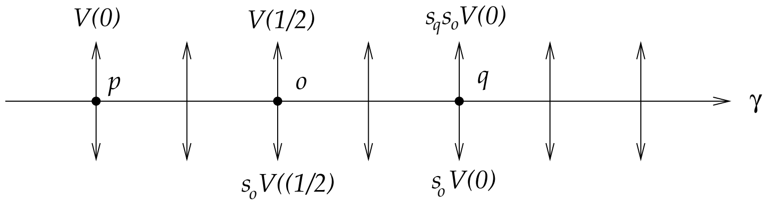
$$-\nabla_a R(b, c)d = \nabla_{-a} R(-b, -c)(-d) = \nabla_a R(b, c)d$$

which shows $\nabla R = 0$.



The easiest example is euclidean space $\mathbb{R}^n$ (left figure); of course $R = 0$ implies $\nabla R = 0$, and we see also the symmetry s_p (point reflection at p). Similar, the sphere $\mathbb{S}^n$ is symmetric where the symmetry at $p \in \mathbb{S}^n$ is the reflection at the line $\mathbb{R}p$ (figure at center). A more interesting example is the *Grassmannian* $\mathbb{G}_k(\mathbb{R}^n)$, the set of all k-dimensional subspaces of $\mathbb{R}^n$. The symmetry at any $E \in \mathbb{G}_k(\mathbb{R}^n)$ is the reflection s_E, the linear map with $s_E = I$ on E and $s_E = -I$ on $E^\perp$, applied to k-dimensional subspaces (right figure). The Riemannian metric on $\mathbb{G}_k(\mathbb{R}^n)$ comes from its embedding into the euclidean vector space S_n of symmetric real matrices by the map $E \mapsto s_E$ (see [7] for more details).

The characterization of symmetric spaces by symmetries is much easier to handle than the complicated PDE $\nabla R = 0$. From now on, a (globally) *symmetric space* is a Riemannian manifold P such that at each point $p \in P$ there is an isometry $s_p : P \to P$ (called symmetry) fixing p and reflecting all geodesics through p. In particular, the isometry group G on P acts transitively: For any $p, q \in P$ there is an isometry f on P with $f(p) = q$. In fact, we may choose a geodesic γ with $\gamma(0) = p$ and $\gamma(1) = q$ and let $o = \gamma(\frac{1}{2})$ be its midpoint. Then $s_o(p) = q$ since s_o reflects γ at $\gamma(\frac{1}{2})$, that is $s_o\gamma(t) = \gamma(1-t)$. We can yet improve this isometry by choosing $f = s_q s_o$ which translates γ (two-fold reflection): $f(\gamma(t)) = \gamma(t+1)$. Moreover, if $V(t)$ is any parallel vector field along γ then $fV(t)$ is again parallel along γ (isometries preserve parallelity) and arises from $V(t)$ through a twofold reflection (by s_o and s_p) which is a translation along γ, hence $fV(t) = V(t+1)$. Such compositions of two symmetries are called *transvections*; they replace the translations in euclidean space. Thus parallel transport along γ happens by isometries.



The transvections along γ form a one-parameter subgroup, a group homomorphism $\phi : \mathbb{R} \to G$, and γ is an orbit of ϕ. Orbits have no self intersection, thus geodesic loops are closed, smooth mappings on $\mathbb{S}^1$.

There is a representation of P as a *coset space* of the isometry group of P or any subgroup G acting transitively on P. Often we choose G to be the group generated by all transvections ("transvection group"). If we fix some point $o \in P$ ("origin") and let $K = \{g \in G : go = o\}$ be the *stabilizer* or *isotropy group* at p, we have an equivariant smooth map $\pi : G \to P$, $g \mapsto go$. The fibres (preimages of points) are cosets gK, $g \in G$. In fact, if $g, h \in G$, then $\pi(g) = \pi(h) \iff go = ho \iff g^{-1}h \in K \iff h \in gK$. Thus π induces an equivariant diffeomorphism $\phi : G/K \to P$, $\phi(gK) = go$, where G/K is the coset space $G/K = \{gK : g \in G\}$. In the example of the Grassmannian $P = \mathbb{G}_k(\mathbb{R}^n)$, we choose the "origin" to be the standard subspace $\mathbb{R}^k \subset \mathbb{R}^n$. The group G may be O_n, then $K = O_k O_{n-k}$ (this is the set of all orthogonal matrices preserving $\mathbb{R}^k$, the block diagonal matrices $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ with $A \in O_k$ and $B \in O_{n-k}$).

The action of the isotropy group K at o commutes with the symmetry, $s_o k = k s_o$ for all $k \in K$. In fact more is true: Conjugation by s_o is an automorphism $\sigma : G \to G$, $\sigma(g) = s_o g s_o$ (recall $s_o^{-1} = s_o$), and K is essentially equal to the fixed group $\hat{K} = \{g \in G : \sigma(g) = g\}$.[1] Vice versa, if G is a Lie group with an involution σ such that $K = \mathrm{Fix}(\sigma)$ is a compact subgroup, then G/K is a symmetric space, and the symmetry s_o at the base point $o = eK$ is induced by σ, namely $s_o(gK) = \sigma(g)K$. Thus symmetric spaces are essentially pairs (G, σ) where G is a Lie group with involution σ such that $K = \mathrm{Fix}\,\sigma$ is compact.

[1] more precisely, $\hat{K}^o \subset K \subset \hat{K}$ where $\hat{K}^o \subset \hat{K}$ denotes the connected component of the unit element of $\hat{K}$; in particular $K^o = \hat{K}^o$.

Moreover, σ acts as a linear involution on the tangent space $T_eG =: \mathfrak{g}$ at the unit element e, the *Lie algebra* of G. Thus we have a decomposition of $\mathfrak{g}$ it into (± 1)-eigenspaces, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The fixed space $\mathfrak{k}$ is the Lie algebra of the fixed group K while the anti-fixed space $\mathfrak{p}$ can be viewed as tangent space of $P = G/K$ at the base point eK. It turns out that in this representation, the curvature tensor R is just (up to sign) the double Lie bracket,[2] $R(x, y)z = \pm[[x, y], z]$ for all $x, y, z \in \mathfrak{p} \subset \mathfrak{g}$. Geometry and algebra match perfectly. The isotropy group K acts linearly both on $\mathfrak{k}$ and $\mathfrak{p}$; the action on $\mathfrak{p}$ is called *isotropy representation* of P (see Section 4).

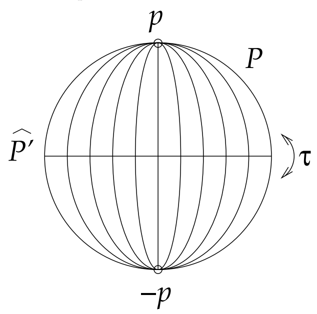
3. Bott Periodicity and Generalizations

One of the classical theorems where geometry and topology are closely related is Bott's periodicity theorem:

> *The sequence of homotopy groups $\pi_k(G)$ is 2-periodic for the unitary group $G = U_n$ and 8-periodic for the orthogonal and symplectic groups $G = O_n$ and $G = Sp_n$, where $n >> k$.*

Recall that $\pi_k(G)$ is the set of homotopy classes of continuous maps from the k-sphere to G. Simply speaking, algebraic topology measures the number and dimension of "holes" in a space. If you have an uncovered hole, you can walk around (mapping a circle, a 1-sphere $\mathbb{S}^1$ into the space) but not through it. Thus holes are measured by "uncovered" maps of a sphere into the space, where "uncovered" means that you cannot contract the sphere into a point.

The proof [16] relies on a construction of chains of certain totally geodesic subspaces of G (which is a symmetric space, as is any compact Lie group). The building blocks for these chains arise from a simple geometric idea: Let P be a symmetric space, p and $-p$ two points in P with the same symmetry,[3] $s_{-p} = s_p$. Let $\Omega^o(P)$ be the set of shortest geodesics γ connecting p to $-p$, say $\gamma(0) = p$ and $\gamma(1) = -p$. Each $\gamma \in \Omega^o(P)$ is uniquely determined by its *midpoint* $\gamma(1/2)$. Thus $\Omega^o(P)$ can be viewed as the subset $\hat{P}' \subset P$ of midpoints of $\gamma \in \Omega^o(P)$. This is the fixed set of an isometry, the reflection[4] $\tau = -s_p$.



Any connected component $P' \subset \hat{P}'$ is called a "centriole". This is *totally geodesic*, that is: every geodesic in P' is also a geodesic in P. This follows from the local uniqueness of geodesics, see figure:

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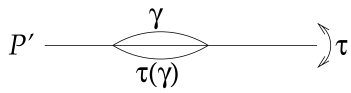
[2]If G is a matrix group, $G \subset GL_n$, then $\mathfrak{g} = T_eG$ also consists of matrices and the Lie bracket is just the commutator of these matrices. In the general case we have to use the Lie derivative of left invariant vector fields on G (which are in 1:1 correspondence to tangent vectors at e). Anyway the Lie bracket is an algebraic object connected to the group structure of G.

[3]Then the point $-p$ is called a *pole* of p, like north and south poles in a sphere. Many compact symmetric spaces P (spheres, Grassmannians, classical groups etc.) allow a nice embedding into euclidean space; if they are invariant under $-I = \mathrm{diag}(-1, \dots, -1)$, then $-p$ is a pole for p for any $p \in P$.

[4]The quotient space $\bar{P} = P/\{\pm I\}$ is again a symmetric space, hence $\gamma : p \rightsquigarrow -p$ becomes a closed geodesic $\bar{\gamma}$ which shows in turn that γ can be extended to a closed geodesic $\gamma : [-1, 1] \to P$ which is invariant under $-I$. In particular, $\gamma(-\frac{1}{2}) = -\gamma(\frac{1}{2})$, but also $\gamma(-\frac{1}{2}) = s_o\gamma(\frac{1}{2})$. Thus $\hat{P}'$ is the fixed set of $\tau = -s_p$.



Moreover, for certain pairs (P, P') and $k < d$ for some d, cf. [17], this has the topological implication

$$\pi_k(P) = \pi_{k-1}(P') \tag{2}$$

In fact, by definition we have $\pi_k(P) = \pi_{k-1}(\Omega P)$ where ΩP denotes the space of paths in P with end points p and $-p$. Using Morse theory,[5] ΩP can be replaced by the space $\Omega^o P$ of shortest geodesics joining p and $-p$.

Example. Let $G \subset GL(\mathbb{R}^n)$ be a connected matrix Lie group with $-I \in G$. This is a symmetric space; the symmetry at the unit element I is $s_I(g) = g^{-1}$. We choose $p = I$ and $-p = -I$. A geodesic $\gamma : \mathbb{R} \to G$ with $\gamma(0) = I$ is just a group homomorphism, $\gamma(s + t) = \gamma(s)\gamma(t)$. If $\gamma(1) = 0$, then the midpoint $J = \gamma(\frac{1}{2})$ satisfies $J^2 = -I$ ("complex structure"). Thus the midpoint set is the set of complex structures in G.

Starting with $P_0 = G$, we use the above construction repeatedly and obtain a chain[6]

$$G \supset P_1 \supset P_2 \supset \cdots \supset P_s \tag{3}$$

For low k [17, 18] this has the topological implication

$$\pi_k(G) = \pi_{k-1}(P_1) = \pi_{k-2}(P_2) = \cdots = \pi_{k-s}(P_s) \tag{4}$$

For $G = G_n := Sp_n, U_n, SO_n$, these chains are given in no. 2,3,4 of Table 1 below. The periodicity now follows just from the observation

$$P_2(U_n) = U_{n/2}, \quad P_4(SO_n) = Sp_{n/8}, \quad P_4(Sp_n) = SO_{n/2} \tag{5}$$

and from the fact that $\pi_k(G_n)$ does not depend on n for $k << n$ since $\mathbb{S} = G_n/G_{n-1}$ is a high dimensional sphere with $\pi_k(\mathbb{S}) = 0$.

By similar methods we can also prove and even generalize Atiyah's version of the periodicity theorem on vector bundles over compact manifolds, see [8].

Peter Quast [18] has classified all chains (3) with length $s \geq 3$. The chains 2, 3, 4 occur in Milnor's book [16] while 1 and 5 are new.[7]

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[5] We use the negative gradient of the energy function $E(\gamma) = \int_I |\gamma'|^2$ on the path space ΩP. The flow of $-\nabla E$ sweeps each $\gamma \in \Omega P$ into some critical point of E which is a geodesic. Most of ΩP is swept to the set of minima, the shortest geodesics. For certain P, all nonminimal geodesics have high index where the index of a geodesic is the number of linear independent deformations decreasing the energy. This phenomenon can be seen already on the sphere $\mathbb{S}^n$: A nonminimal geodesic between north and south poles must wrap around the sphere at least once and can be shortened in any of the $n - 1$ perpendicular directions. Hence the domains of attraction for nonminimal geodesics have high codimension d and can be avoided by any sphere of dimension $< d$ in $\Omega(G)$. Thus $\pi_j(\Omega P) = \pi_j(\Omega^o P)$ for any $j < d$.

[6] By the above example, all elements of P_k, $k \geq 1$ are complex structures in G. In P_k we have to choose $p = J_k$ and $-p = -J_k$ such that $J_1, \ldots, J_k$ anticommute. A system of anticommuting complex structures is the matrix representation of a Clifford algebra.

[7] In chain 1, SO'_{4n} denotes the half spin representation of $Spin_{4n}$; like SO_{4n} it arises from $Spin_{4n}$ by dividing out a central $\mathbb{Z}_2$, but it is the other $\mathbb{Z}_2$-factor in the center $\mathbb{Z}_2 \times \mathbb{Z}_2$ of $Spin_{4n}$. The most interesting case is SO'_{12}, being part of the isotropy representation of the Rosenfeld plane $\mathbb{OHP}^2$, see next section and table 2 below. The representation is quaternionic which allows an interpretation of the P_r as "Grassmannians", see 4.2 in [18]. Chain 5 is interesting since it relates the octonionic projective plane $\mathbb{OP}^2$ to the exceptional group E_7. It can be viewed as the octonionic analogue of the classical chains 2, 3, 4. The other projective planes $\mathbb{RP}^2$, $\mathbb{CP}^2$, $\mathbb{HP}^2$ are the end points of these chains for $n = 3$ and $p = 1$. These four groups $G = Sp_3$, U_6, SO'_{12}, E_7 have more in common: They belong to the isotropy group (s-representation) of the exceptional quaternionic symmetric spaces $S = F_4/Sp_3Sp_1$, E_6/SU_6Sp_1, $E_7/SO'_{12}Sp_1$, E_8/E_7Sp_1 respectively, see Table 2 at the end of the paper. This turns G into a quaternionic matrix group, similar to chain No. 2 (starting with Sp_n) where P_1 is the set of totally complex subspaces, P_2 the set of totally real subspaces and P_3 a real Grassmannian. Similarly, we may interpret the subspaces P_i for the $\mathbb{KP}^2$-chains as sets of of certain totally geodesic subspaces of S, cf. [18], Section 4.2.

No.	G	P_1	P_2	P_3	P_4
1	SO'_{4n}	$G_2^+(\mathbb{R}^{4n})$	$(\mathbb{S}^1\mathbb{S}^{4n-3})/\pm$	$\mathbb{S}^{4n-4}$	$\mathbb{S}^{4n-5}$ if $n \geq 2$
2	Sp_n	Sp_n/U_n	U_n/SO_n	$G_p(\mathbb{R}^n)$	$SO_{n/2}$ if $p = n/2$
3	U_{2n}	$G_n(\mathbb{C}^{2n})$	U_n	$G_p(\mathbb{C}^n)$	$U_{n/2}$ if $p = n/2$
4	SO_{4n}	SO_{4n}/U_{2n}	U_{2n}/Sp_n	$G_p(\mathbb{H}^n)$	$Sp_{n/2}$ if $p = n/2$
5	E_7	E_7/U_1E_6	U_1E_6/F_4	$G_1(\mathbb{O}^3) = \mathbb{OP}^2$	-

Table 1

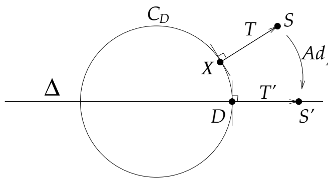
4. Normal forms for matrices

Let us start with a famous theorem in undergraduate linear algebra:

A real symmetric matrix S has an orthonormal basis of eigenvectors.

The proof is not quite obvious. It is easy to see that eigenspaces of a symmetric matrix are perpendicular, due to $S^T = S$, but why are the eigenvalues real? This needs some analysis. One may use that a symmetric matrix $T = (t_{ij})$ is positive (negative) definite if and only if all its principal minors $\det(t_{ij})_{i,j\leq k}$, $k = 1, \dots, n$, are positive (negative and positive in turn). For large $\lambda \in \mathbb{R}$, the matrix $S_\lambda = \lambda I - S$ is positive definite and all its principal minors are positive. By decreasing λ we arrive at some value λ_o where first time one of the principal minor vanishes; this must happen because S_λ will become negative definite as $\lambda \to -\infty$, hence every second minor must change its sign. This λ_o is the largest eigenvalue of S.

Let us discuss another proof using some differential geometry on the space $\mathcal{S}_n$ of real symmetric $n \times n$-matrices. This is a euclidean vector space with inner product $\langle X, Y\rangle = \operatorname{trace} XY$. We want to show that each conjugacy class in $\mathcal{S}_n$ intersects the set of diagonal matrices $\Delta \subset \mathcal{S}_n$. Conjugacy classes are orbits of the group SO_n acting linearly on $\mathcal{S}_n$ by $S \mapsto \operatorname{Ad}_A S := ASA^T$ (for any $A \in SO_n$, $S \in \mathcal{S}_n$). Choose some $D \in \Delta$ with distinct diagonal entries. Let $C_D = \{\operatorname{Ad}_A D : A \in SO_n\}$ be the orbit of D under this action (the conjugacy class of D). This is a smooth compact submanifold of $\mathcal{S}_n$, and an easy calculation[8] shows that Δ is precisely the normal space of C_D at D. Now let $S \in \mathcal{S}_n$ be arbitrary. Then there is a point $X \in C_D$ which is closest to S. Thus $T = S - X$ is in the normal space of C_D at X (otherwise X could not be closest to S). Since $X \in C_D$, there is $A \in SO_n$ with $\operatorname{Ad}_A X = D$, and since Ad_A preserves the orbit C_D and the inner product on $\mathcal{S}_n$, it maps the normal vector T at X onto a normal vector T' at D. Thus $T' \in \Delta$ and hence $S' := \operatorname{Ad}_A S = \operatorname{Ad}_A(X + T) = D + T' \in \Delta$ which proves the theorem.



The main ingredience for this proof is the fact that Δ is the common normal space for all principal orbits through Δ. Such a linear action is called *polar*: there is a subspace Δ (called "section") which meets every orbit and each time the intersection is perpendicular. An intersection point of an orbit with Δ is called a *normal form* for the elements in the orbit. The normal form of a real symmetric matrix S is a diagonal matrix D which is conjugate to S; its entries are the eigenvalues of S. The same holds for hermitian matrices over the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and even the octonions $\mathbb{O}$ when $n \leq 3$. While the usual linear algebra proof breaks down for $\mathbb{H}$ and $\mathbb{O}$ since there is no obvious way to define determinants, the "new" proof works in these cases as well.

[8] Let $\mathcal{A}_n$ be the space of antisymmetric matrices, the Lie algebra of SO_n. Then $T_DC_D = [\mathcal{A}_n, D]$ and hence $Y \in (T_DC_D)^\perp \iff 0 = \langle[\mathcal{A}_n, D], Y\rangle = \langle\mathcal{A}_n, [D, Y]\rangle \iff [D, Y] = 0$ (since $[\mathcal{S}_n, \mathcal{S}_n] \subset \mathcal{A}_n$) $\iff Y \in \Delta$.

A theorem of J. Dadok [5, 9] says that polar representations are essentially[9] in one-to-one correspondence to compact symmetric spaces, being the isotropy representations of such spaces, so called s-representations.[10] The case of symmetric and hermitian matrices corresponds to the symmetric spaces with Dynkin diagram of type A which are SU_n/SO_n, SU_n, SU_{2n}/Sp_n, and E_6/F_4 [12, p. 532 ff].[11]

There are several similar normal form problems in linear algebra. One is the "singular value decomposition" which assigns a rectangular diagonal matrix to an arbitrary $(k \times n)$-matrix using the natural action of $SO_k \times SO_n$ on $\mathbb{R}^{k\times n}$. This is a polar representation; the corresponding symmetric space is the Grassmannian $G_k(\mathbb{R}^{n+k})$. A more exotic example is the (half) spin representation of $Spin_{8+k}$ on the vector space $(\mathbb{O} \otimes_{\mathbb{R}} \mathbb{K})^2$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and $k = 1, 2, 4, 8$; here the section is spanned by $e_j \otimes e_j$ for $j = 1, \dots, k$ where $e_1, \dots, e_k$ form the canonical $\mathbb{R}$-basis of $\mathbb{K} \subset \mathbb{O}$, [6]. These polar actions correspond to the so called Rosenfeld planes which I am going to explain in the next section.

On the first glance, the normal form problem seems a very local, even infinitesimal approach to symmetric spaces: only the tangent space of a single point is involved. However, it is related to global problems in various ways. First, some compact symmetric spaces are linked to s-representations yet differently: they occur among their orbits. E.g. in our first example (real symmetric matrices), the orbits of diagonal matrices with just two different eigenvalues are Grassmannians. Thus we obtain very nice (so called *extrinsic symmetric*) embeddings of symmetric spaces into euclidean space which are extremely helpful in order to understand the topology of these spaces, using Morse theory of the height functions of the ambient space. Among the compact symmetric spaces, all the classical and 4 of the 17 exceptional ones can be embedded in this way (up to coverings and $\mathbb{S}^1$-factors).

Further, the isotropy representation of a symmetric space $P = G/K$ is just the linearization of the *isotropy action* of $K \subset G$ on P itself. This brings us from polar *representations* to polar *actions* which have been classified by Andreas Kollross [14]. The normal form problem for orthogonal and unitary matrices belongs to that area. We can relate isotropy representation and action by the map $\mathfrak{p} \to P$, $v \mapsto \gamma_v(t)$ for fixed t where γ_v is the geodesic with $\gamma_v(0) = o$ and $\gamma_v'(0) = v$ It is interesting what happens to an extrinsic symmetric s-orbit $M \subset \mathfrak{p}$ under this map. We obtain a one-parameter family of embeddings $M \subset P$ starting and ending at a fixed point of K with a totally geodesic embedding in the middle. E.g. the Grassmannians occur as mid point sets of shortest geodesics in SU_n joining the identity matrix I to any other element of the center $\mathbb{Z}_n \subset SU_n$. We have seen such examples in section 3.

5. Symmetric spaces and division algebras

There are precisely four normed real division algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. These are closely connected to symmetric spaces. The "classical" symmetric spaces are related to the linear algebra over the associative division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and come in 7 infinite chains of increasing dimension:

(1) Grassmannians: $\{\mathbb{K}^p \subset \mathbb{K}^n\} = G_p(\mathbb{K}^n)$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$,

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[9] The notion of polarity involves only the orbits, not the group action. Sometimes there are subgroups acting with the same orbits; we must consider the largest possible group acting effectively with these orbits.

[10] If $\rho : K \to O(V)$ is an s-representation, the corresponding symmetric space is G/K where $\mathfrak{g} = \mathfrak{k} \oplus V$ with the Lie bracket $[A, X] \in V$, $[X, Y] \in \mathfrak{k}$ for $A \in \mathfrak{k}$, $X, Y \in V$ given by $[A, X] = \rho(A)X$ while $\langle A, [X, Y]\rangle_{\mathfrak{k}} = \langle \rho(A)X, Y\rangle$ for $\langle A, B\rangle_{\mathfrak{k}} = -\operatorname{trace}(\rho(A)\rho(B)) - \operatorname{trace}(\operatorname{ad}(A)\operatorname{ad}(B))$.

[11] The spaces with Dynkin diagram A_3 are SU_3/SO_3, SU_3, SU_6/Sp_3, E_6/F_4. They played a key role in my mathematical life: The only known infinite families of compact simply connected positively curved Riemannian manifolds are the orbit spaces of isometric circle actions on SU_3 and SU_6/Sp_3 (with a non-symmetric metric); the first family has been discovered by Aloff, Wallach and myself, the second one by Berger and Bazaikin. There are only few irreducible symmetric spaces which allow free isometric circle actions: compact Lie groups, SU_{2n}/SO_{2n}, SU_{2n}/Sp_n and the Grassmannians $G_{2k-1}(\mathbb{R}^{2n})$, cf. [11]. In particular, E_6/F_4 does not allow such actions as was proved first by Robert Bock [2].

(2) $\mathbb{R}$-structures on $\mathbb{C}^n$: $\{\mathbb{R}^n \subset \mathbb{C}^n\} = U_n/SO_n$
$\mathbb{C}$-structures on $\mathbb{H}^n$: $\{\mathbb{C}^n \subset \mathbb{H}^n\} = Sp_n/U_n$,
(3) $\mathbb{C}$-structures on $\mathbb{R}^{2n}$: $\{\mathbb{R}^{2n} \cong \mathbb{C}^n\} = SO_{2n}/U_n$
$\mathbb{H}$-structures on $\mathbb{C}^{2n}$: $\{\mathbb{C}^{2n} \cong \mathbb{H}^n\} = U_{2n}/Sp_n$.

On the other hand, the non-associative division algebra $\mathbb{O}$ seems to be related to the finitely many "exceptional" symmetric spaces, but this relation is not yet fully understood. Let us restrict our attention to irreducible type-I symmetric spaces (G/K with G compact and simple). This class consists of the classical spaces (Grassmannians and $\mathbb{R}, \mathbb{C}, \mathbb{H}$-structures, see above) together with 12 exceptional spaces. These include the *Rosenfeld planes* with dimension $16, 32, 64, 128$ which seem to continue the series of classical projective planes $\mathbb{KP}^2$ of dimensions $2, 4, 8, 16$. Boris Rosenfeld in 1956 tried to define them as projective planes $\mathbb{AP}^2$ over the non-associative algebra $\mathbb{A} = \mathbb{O} \otimes_{\mathbb{R}} \mathbb{K} =: \mathbb{OK}$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively. Though this was not successful [1], Rosenfeld's idea somehow survived, and following Besse [3] we keep calling these spaces Rosenfeld planes $\mathbb{AP}^2$. In several aspects, they behave as if they were projective planes over $\mathbb{A}$.

(1) There are "projective lines" $\mathbb{AP}^1 \subset \mathbb{AP}^2$ which are oriented Grassmannians[12] $G_k^+(\mathbb{R}^{8+k})$ where $k = \dim_{\mathbb{R}} \mathbb{K}$. However, the intersection of these "lines" is not always transversal. But we still have the duality $\mathbb{AP}^2 \cong \{\mathbb{AP}^1 \subset \mathbb{AP}^2\}$ where the latter space denotes the dual plane, that is the set of all "lines" $\mathbb{AP}^1$ in $\mathbb{AP}^2$.
(2) While the isotropy representation of $\mathbb{KP}^2$ is essentially[13] the (half) spin representation of $Spin_{1+k}$ on $\mathbb{K}^2$, the isotropy representation of $\mathbb{AP}^2$ is essentially the (half) spin representation for $Spin_{8+k}$ on $\mathbb{A}^2$.
(3) The Lie algebra of the isometry group of $\mathbb{KP}^2$ can be described in terms of tracefree anti-hermitian 3×3-matrices over $\mathbb{K}$, and this remains true for $\mathbb{AP}^2$, replacing $\mathbb{K}$ by $\mathbb{A}$ (Vinberg's formula [1, p. 192]).[14]

All other type-I exceptional symmetric spaces (except G_2/SO_4, the space of all quaternion type subalgebras of the octonions) are obtained as spaces of self-reflective subspaces[15] of the Rosenfeld planes. If we believe in Rosenfeld's description as $\mathbb{AP}^2$, we can conjecture that these subspaces are projective subplanes $\mathbb{BP}^2 \subset \mathbb{AP}^2$ (similar to $\mathbb{RP}^2 \subset \mathbb{CP}^2$), where $\mathbb{B} \subset \mathbb{A}$ is a *selfreflective subalgebra*.[16] E.g. E_6/F_4 can be viewed as the set of all totally geodesic embeddings of $\mathbb{OP}^2$ into $\mathbb{OCP}^2$ (cf. [3], p. 313); we will write briefly $E_6/F_4 = \{\mathbb{OP}^2 \subset \mathbb{OCP}^2\}$. Here $\mathbb{A} = \mathbb{OC}$ and $\mathbb{B} = \mathbb{OR} = \mathrm{Fix}(\rho)$ with $\rho = \mathrm{id} \otimes \kappa_2$ on $\mathbb{A}$ where κ_2 is complex conjugation in the second tensor factor $\mathbb{C}$. But other cases, like E_6/Sp_4 and E_7/SU_8, are less obvious. It seems that these correspond to involutions ρ of type (b) on $\mathbb{A}$, see footnote 16.

An analogous problem for classical spaces has been solved in a common paper with Somayeh Hosseini [10], based on recent work of Y. Huang and N.C. Leung [13]:

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[12] There is a small deviation for $\mathbb{A} = \mathbb{OO}$: According to [4] $\mathbb{OOP}^2 = G_8^{\#}(\mathbb{R}^{16})$ rather than $G_8^+(\mathbb{R}^{16})$ (which would be a two-fold covering of $G_8^{\#}(\mathbb{R}^{16})$).

[13] As the spin representation of $Spin'_{2n}$ is complex for odd n and quaternional for even n, we have to add the U_1 and Sp_1 factors in order to obtain the full isotropy representation.

[14] $\mathfrak{g} = \mathrm{Der}_\kappa(\mathbb{A}) \oplus A_o(\mathbb{A}, 3)$ where $\mathrm{Der}_\kappa(\mathbb{A})$ denotes the derivations of $\mathbb{A}$ which commute with the conjugation $\kappa(x \otimes y) = \bar{x} \otimes \bar{y}$ and where $A_o(\mathbb{A}, 3)$ denotes the anti-hermitian trace-zero 3×3-matrices over $\mathbb{A}$. The Lie bracket is more complicated, cf [1].

[15] A reflective submanifold Q of a symmetric space P is a connected component of the fixed set of some isometric involution τ on P. Reflective submanifolds come in pairs: For any $q \in Q$ there is another reflective submanifold Q' through q perpendicular to Q which is a fixed set component of the involution $\tau \circ s_q$ of P (where s_q denotes the symmetry at q). If Q and Q' are congruent, the submanifold is called self-reflective. For any reflective submanifold $Q \subset P$, the set of all $Q' \subset P$ with Q' congruent to Q is again a symmetric space called $\{Q \subset P\}$; its symmetry at Q is τ.

[16] This is the fixed subalgebra (1-eigenspace) of an involution ρ of $\mathbb{A}$ with eigenspaces of equal dimensions and such that ρ commutes with the conjugation κ on $\mathbb{A}$ ("balanced involution"). There are two kinds of such involutions on $\mathbb{A} = \mathbb{K} \otimes \mathbb{L}$: (a) $\rho = \sigma \otimes \mathrm{id}$ or $\mathrm{id} \otimes \tau$ and (b) $\rho = \sigma \otimes \tau$ where σ, τ are balanced involutions on $\mathbb{K}, \mathbb{L}$, respectively. In case (b), the subalgebra $\mathbb{B}$ is a tensor product with the paracomplex numbers $\mathcal{C} = \mathbb{R} \oplus \mathbb{R}s$ with $s^2 = 1$.

Theorem 1. *All classical type-I symmetric spaces (up to coverings and S^1 factors) are either Grassmannians $G_p(\mathbb{A}^n)$ or inclusion sets $\{G_p(\mathbb{B}^n) \subset G_p(\mathbb{A}^n)\}$ where $\mathbb{B}$ is some self-reflective subalgebra of $\mathbb{A}$ and $\mathbb{A} = \mathbb{K} \otimes \mathbb{L} =: \mathbb{KL}$ with $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ or $\mathbb{A}$ itself is already a self-reflective subalgebra of $\mathbb{K} \otimes \mathbb{L}$. Some symmetric spaces allow several such descriptions, cf. Table 2.*[17]

No.	Type	Space	dim	rk	$\mathbb{B}$	$\mathbb{A}$	Grassmannian
1	AI	U_n/O_n	$\frac{n(n+1)}{2}$	n	$\mathbb{R}$	$\mathbb{C}$	$\{\mathbb{RP}^{n-1} \subset \mathbb{CP}^{n-1}\}$
2		U_{2n}/O_{2n}	$n(2n+1)$	$2n$	$\tilde{\mathbb{C}}\mathbb{C}$	$\mathbb{HC}$	$\{G_2(\mathbb{R}^{2n}) \subset G_2(\mathbb{C}^{2n})\}$
3	AII	U_{2n}/Sp_n	$2n(n-1)$	n	$\mathbb{H}$	$\mathbb{HC}$	$\{\mathbb{HP}^{n-1} \subset G_2(\mathbb{C}^{2n})\}$
4	$AIII$	$U_{p+q}/(U_pU_q)$	$2pq$	p		$\mathbb{C}$	$G_p(\mathbb{C}^{p+q})$
5		$U_{2n}/(U_{2p}U_{2n-2p})$	$8p(n-p)$	$2p$		$\mathbb{HC}$	$G_p(\mathbb{HC}^n)$
6		$U_{2n}/(U_nU_n)$	$2n^2$	n	$\mathbb{CC}$	$\mathbb{HC}$	$\{(\mathbb{CP}^{n-1})^2 \subset G_2(\mathbb{C}^{2n})\}$
7	BDI	O_{p+q}/O_pO_q	pq	p		$\mathbb{R}$	$G_p(\mathbb{R}^{p+q})$
8		$O_{4n}/O_{4p}O_{4n-4p}$	$16p(n-p)$	$4p$		$\mathbb{HH}$	$G_p(\mathbb{HH}^n)$
9		O_{2n}/O_nO_n	n^2	n	$\mathcal{C}$	$\tilde{\mathbb{C}}\mathbb{C}$	$\{(\mathbb{RP}^{n-1})^2 \subset G_2(\mathbb{R}^{2n})\}$
10		$O_{4n}/O_{2n}O_{2n}$	$4n^2$	$2n$	$\tilde{\mathbb{C}}\mathbb{CC}$	$\mathbb{HH}$	$\{G_2(\mathbb{R}^{2n})^2 \subset G_4(\mathbb{R}^{4n})\}$
11	$DIII$	O_{2n}/U_n	$n(n-1)$	$[\frac{n}{2}]$	$\mathbb{C}$	$\tilde{\mathbb{C}}\mathbb{C}$	$\{\mathbb{CP}^{n-1} \subset G_2(\mathbb{R}^{2n})\}$
12		O_{4n}/U_{2n}	$2n(2n-1)$	n	$\mathbb{HC}$	$\mathbb{HH}$	$\{G_2(\mathbb{C}^{2n}) \subset G_4(\mathbb{R}^{4n})\}$
13	CI	Sp_n/U_n	$n(n-1)$	n	$\mathbb{C}$	$\mathbb{H}$	$\{\mathbb{CP}^{n-1} \subset \mathbb{HP}^{n-1}\}$
14	CII	Sp_{p+q}/Sp_pSp_q	$4pq$	p		$\mathbb{H}$	$G_p(\mathbb{H}^{p+q})$
15	EI	E_6/Sp_4	42	6	$\hat{\mathbb{C}}\mathbb{H}$?	$\mathbb{OC}$	$\{G_2(\mathbb{H}^4)/\mathbb{Z}_2 \subset \mathbb{OCP}^2\}$
16	EII	E_6/SU_6Sp_1	40	4	$\mathbb{HC}$	$\mathbb{OC}$	$\{G_2(\mathbb{C}^6) \subset \mathbb{OCP}^2\}$
17	$EIII$	$E_6/Spin_{10}U_1$	32	2		$\mathbb{OC}$	$\mathbb{OCP}^2$
18	EIV	E_6/F_4	26	2	$\mathbb{O}$	$\mathbb{OC}$	$\{\mathbb{OP}^2 \subset \mathbb{OCP}^2\}$
19	EV	E_7/SU_8	70	7	$\hat{\mathbb{C}}\mathbb{HC}$?	$\mathbb{OH}$	$\{G_4(\mathbb{C}^8)/\mathbb{Z}_2 \subset \mathbb{OHP}^2\}$
20	EVI	$E_7/SO'_{12}Sp_1$	64	4		$\mathbb{OH}$	$\mathbb{OHP}^2$
21					$\mathbb{HH}$	$\mathbb{OH}$	$\{G_4^+(\mathbb{R}^{12}) \subset \mathbb{OHP}^2\}$
22	$EVII$	E_7/E_6U_1	54	3	$\mathbb{OC}$	$\mathbb{OH}$	$\{\mathbb{OCP}^2 \subset \mathbb{OHP}^2\}$
23	$EVIII$	E_8/SO'_{16}	128	8		$\mathbb{OO}$	$\mathbb{OOP}^2$
24					$\hat{\mathbb{C}}\mathbb{HH}$?	$\mathbb{OO}$	$\{G_8^{\#}(\mathbb{R}^{16}) \subset \mathbb{OOP}^2\}$
25	EIX	E_8/E_7Sp_1	112	4	$\mathbb{OH}$	$\mathbb{OO}$	$\{\mathbb{OHP}^2 \subset \mathbb{OOP}^2\}$
26	FI	F_4/Sp_3Sp_1	28	4	$\mathbb{H}$	$\mathbb{O}$	$\{\mathbb{HP}^2 \subset \mathbb{OP}^2\}$
27	FII	$F_4/Spin_9$	16	1		$\mathbb{O}$	$\mathbb{OP}^2$
28	GI	G_2/SO_4	8	2	$\mathbb{H}$	$\mathbb{O}$	$\{\mathbb{H} \subset \mathbb{O}\}$

TABLE 2

The proof is mainly by identifying the group G of orthogonal $\mathbb{A}$-linear maps g on $\mathbb{A}^n$. E.g. if $\mathbb{A} = \mathbb{HC}$, we have on $V = \mathbb{A}^n$ two anticommuting complex structures i and j and another complex structure $\hat{i} = 1 \otimes i$ which commutes with i, j. Then $S = i\hat{i}$ with $S^2 = I$ has i-invariant (± 1)-eigenspaces $V_\pm$ which are interchanged by j. Any $g \in G$ commutes with $i, j, \hat{i}$. Thus g preserves the eigenspaces $V_\pm$ and is already determined by its restriction to V_- since $V_+ = jV_-$. Since g commutes with i, the restriction $g|V_-$ is in the unitary group of $V_- \cong \mathbb{C}^{2n}$ and hence $G = U_{2n}$.

But this proof does not apply to cases where $\mathbb{A}$ is non-associative because then $\mathbb{A}^n$ is no longer an $\mathbb{A}$-module. Therefore in the second part of Table 2 below we have used results of D.S.P. Leung and Chen-Nagano [15, 4] on self-reflective submanifolds. If $Q \subset P = G/K$ is a (self-)reflective subspace, then $\{Q \subset P\} = G/G_Q$, where $G_Q = \{g \in G : g(Q) = Q\}$, and G_Q contains the symmetry group of Q as a normal subgroup. Thus it is easy to identify the spaces $\{Q \subset P\}$. However, the pairs

[17] The second column in Table 2 is the type of the space according to E. Cartan's classification, cf. [12]. In the last column, the space is represented either as a Grassmannian $G_p(\mathbb{A}^n)$ or as a space of inclusions $\{G_p(\mathbb{B}^n) \subset G_p(\mathbb{A}^n)\}$.

$(\mathbb{B}, \mathbb{A})$ given in No. 15, 19 and 24 are only a conjecture, following [13].[18] If we look for a proof, Vinberg's formula [1, p. 192] gives some hope to understand the corresponding groups at least on the infinitesimal level. This requires an extension of Vinberg's formula to self-reflective subalgebras of $\mathbb{O} \otimes \mathbb{K}$ and also to associative algebras in order to allow comparison with the classical case.

6. Conclusion

The linear algebra which is relevant to Riemannian geometry has two parts. Part 1 is the linear algebra over the the associative division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and it is connected to the classical symmetric spaces. Milnor's chains proving Bott's periodicity theorem and relating all these structures (cf. Section 3) seem of fundamental importance for Linear Algebra 1. Part 2 should be a restricted linear algebra over $\mathbb{O}$ which is not yet fully developed. It must be quite different. It cannot contain vectors and modules and linear maps, but projective lines and planes and even (3×3)-matrices seem to survive. As we have seen, Milnor's proof extends to some exceptional spaces, but unfortunately not to all of them. In Section 5 we have tried a different approch, a description of symmetric spaces - classical as well as exceptional - in terms of the algebras $\mathbb{A} = \mathbb{K} \otimes_{\mathbb{R}} \mathbb{L}$ with $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. For the classical spaces (avoiding $\mathbb{O}$) this is a theorem, for the exceptional ones it is a conjecture, but with a lot of evidence and support. So let's try to learn Linear Algebra 2.

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[18] For a subalgebra $\mathbb{B}' \subset \mathbb{A} = \mathbb{O}\mathbb{K}$ we denote by $\mathbb{B} = \hat{\mathcal{C}}\mathbb{B}' \subset \mathbb{A}$ a subalgebra of type $\mathbb{B}' \oplus s\mathbb{B}'$ for a certain paracomplex structure $s \in \mathbb{A} \setminus \mathbb{B}'$; note that this is a non-associative subalgebra of $\mathbb{A}$.