SYMMETRIC SPACES AND DIVISION ALGEBRAS

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Abstract. Working out an idea of Huang and Leung [6, 7] we show that all classical compact symmetric spaces can be represented as sets of subspaces of either one of the following two types: $\{\mathbb{A}^p \subset \mathbb{A}^n\}$ for $p < n$ or $\{\mathbb{B}^n \subset \mathbb{A}^n\}$, where $\mathbb{A}$ is equal or closely related to the tensor algebra $\mathbb{K} \otimes \mathbb{L}$ of two division algebras $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and where $\mathbb{B}$ is a certain half-dimensional subalgebra of $\mathbb{A}$ (joint work with S. Hosseini, [4]). We will discuss a possible extension of this result - at least on the Lie algebra level - to exceptional symmetric spaces where also the octonion algebra $\mathbb{O}$ will show up. This is work in progress.

1. Classical and Exceptional Symmetric spaces

There are precisely four normed real division algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. These are closely connected to Riemannian symmetric spaces. Let us restrict our attention to irreducible symmetric spaces G/K of type I where G compact and simple. The associative division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ correspond to the classical spaces which form seven infinite series,

(1) Grassmannians: $\{\mathbb{K}^p \subset \mathbb{K}^n\} = G_p(\mathbb{K}^n)$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$,
(2) $\mathbb{R}$-structures on $\mathbb{C}^n$: $\{\mathbb{R}^n \subset \mathbb{C}^n\} = U_n/O_n$
$\mathbb{C}$-structures on $\mathbb{H}^n$: $\{\mathbb{C}^n \subset \mathbb{H}^n\} = Sp_n/U_n$,
(3) $\mathbb{C}$-structures on $\mathbb{R}^{2n}$: $\{\mathbb{R}^{2n} \cong \mathbb{C}^n\} = O_{2n}/U_n$
$\mathbb{H}$-structures on $\mathbb{C}^{2n}$: $\{\mathbb{C}^{2n} \cong \mathbb{H}^n\} = U_{2n}/Sp_n$,

while the nonassociative division algebra $\mathbb{O}$ are somehow related to the 12 exceptional spaces, but this connection is not yet fully understood. The latter spaces include the *Rosenfeld planes* with dimensions $16, 32, 64, 128$ which seem to continue the series of classical projective planes $\mathbb{KP}^2$ of dimensions $2, 4, 8, 16$. Boris Rosenfeld in 1956 tried to define those as projective planes $\mathbb{AP}^2$ over the non-associative algebra $\mathbb{A} = \mathbb{O} \otimes_{\mathbb{R}} \mathbb{K} =: \mathbb{OK}$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively. Though this was not successful [1], Rosenfeld's idea somehow survived, and following Besse [2] we keep calling these spaces Rosenfeld planes $\mathbb{AP}^2$. In several aspects, they behave as if they were projective planes over $\mathbb{A}$.

(1) There are "projective lines" $\mathbb{AP}^1 \subset \mathbb{AP}^2$ which are oriented Grassmannians[1] $G_k^+(\mathbb{R}^{8+k})$ with $k = \dim_{\mathbb{R}} \mathbb{K}$. However, the intersection of these "lines" is not always transversal. But we still have the duality $\mathbb{AP}^2 \cong \{\mathbb{AP}^1 \subset \mathbb{AP}^2\}$ where the latter space denotes the "dual projective plane", that is the set of all "lines" $\mathbb{AP}^1$ in $\mathbb{AP}^2$.
(2) While the isotropy representation of $\mathbb{KP}^2 = \mathbb{RKP}^2$ is essentially[2] the (half) spin representation of $Spin_{1+k}$ on $\mathbb{K}^2$, the isotropy representation of $\mathbb{AP}^2 = \mathbb{OKP}^2$ is essentially the (half) spin representation for $Spin_{8+k}$ on $\mathbb{A}^2$.
(3) The Lie algebra of the isometry group of $\mathbb{KP}^2$ can be described in terms of tracefree anti-hermitian 3×3-matrices over $\mathbb{K}$, and this remains true for $\mathbb{AP}^2$, replacing $\mathbb{K}$ by $\mathbb{A}$ (Vinberg's formula [1, p. 192]).[3]

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[1]There is a small deviation for $\mathbb{A} = \mathbb{OO}$: According to [3], $\mathbb{OOP}^2 = G_8^{\#}(\mathbb{R}^{16})$ rather than $G_8^+(\mathbb{R}^{16})$ (which would be a two-fold covering of $G_8^{\#}(\mathbb{R}^{16})$).

[2]As the spin representation of $Spin_{2n}$ is complex for odd n and quaternional for even n, we have to add the U_1 and Sp_1 factors in order to obtain the full isotropy representation.

[3]$\mathfrak{g} = \mathrm{Der}_\kappa(\mathbb{A}) \oplus A_o(\mathbb{A}, 3)$ where $\mathrm{Der}_\kappa(\mathbb{A})$ denotes the derivations of $\mathbb{A}$ which commute with the conjugation $\kappa(x \otimes y) = \bar{x} \otimes \bar{y}$ and where $A_o(\mathbb{A}, 3)$ denotes the anti-hermitian trace-zero 3×3-matrices over $\mathbb{A}$. The Lie bracket is more complicated, cf [1].

2. Selfreflective subspaces and subalgebras

All other type-I exceptional symmetric spaces (except G_2/SO_4, the space of all quaternion type subalgebras of the octonions) are obtained as *spaces of self-reflective subspaces* of the Rosenfeld planes. A *reflective* submanifold Q of a symmetric space P is a connected component of the fixed set of some isometric involution r on P. Reflective submanifolds come in pairs: For any $q \in Q$ there is another reflective submanifold Q' through q perpendicular to Q which is a fixed set component of the involution $r \circ s_q$ of P (where s_q denotes the symmetry at q). If Q and Q' are congruent, the submanifold is called self-reflective. For any reflective submanifold $Q \subset P$, the set of all $Q' \subset P$ with Q' congruent to Q is again a symmetric space called $\{Q \subset P\}$; its symmetry at Q is r.

If we believe in Rosenfeld's description as $\mathbb{AP}^2$, we can conjecture that these self-reflective subspaces are either "projective lines" $\mathbb{AP}^1 \subset \mathbb{AP}^2$ or projective subplanes $\mathbb{BP}^2 \subset \mathbb{AP}^2$ (like $\mathbb{RP}^2 \subset \mathbb{CP}^2$), where $\mathbb{B} \subset \mathbb{A}$ is a *selfreflective subalgebra.* This is the fixed subalgebra (1-eigenspace) of an involution ρ of $\mathbb{A}$ which is "balanced" in the sense that the two eigenspaces have equal dimensions and that ρ commutes with the conjugation κ on $\mathbb{A}$ (with $\kappa(x \otimes y) = \bar{x} \otimes \bar{y}$). There are the following two kinds of such involutions on $\mathbb{A} = \mathbb{K} \otimes \mathbb{L}$.

(a) $\rho = \sigma \otimes \mathrm{id}$ or $\rho = \mathrm{id} \otimes \tau$,
(b) $\rho = \sigma \otimes \tau$

where σ, τ are balanced involutions on $\mathbb{K}$, $\mathbb{L}$, respectively. In both cases we have $\mathbb{A} = \mathbb{B} + u\mathbb{B}$ for some $u \in \mathbb{A}$ which belongs to the (-1)-eigenspace of ρ. In some cases, the subalgebra $\mathbb{B}$ is a tensor product with the paracomplex numbers $\mathcal{C} = \mathbb{R} \oplus \mathbb{R}s$ with $s^2 = 1$ where s anticommutes with the other generators. The possible choices are listed in Table 1 where $\hat{?}$ always refers to the second tensor factor, κ and κ_p denote the conjugations in $\mathbb{C}$ and $\mathcal{C}$, respectively, and τ is the automorphism of $\mathbb{O}$ fixing $\mathbb{H}$ which corresponds to $-I \in SO_4 \subset G_2$.

No.	$\mathbb{A}$	generators	ρ	Case	u	$\mathbb{B}$	generators
1	$\mathbb{C}$	i	κ	a	i	$\mathbb{R}$	-
2	$\mathbb{H}$	i, j	$Ad(i)$	a	j	$\mathbb{C}$	i
3	$\mathbb{CC}$	$i, \hat{i}$	κ	a	i	$\mathbb{C}$	i
4	$\mathbb{CC}$	$i, \hat{i}$	$\kappa\hat{\kappa}$	b	$i, \hat{i}$	$\mathcal{C}$	$i\hat{i}$
5	$\mathbb{HC}$	$i, j, \hat{i}$	$\hat{\kappa}$	a	$\hat{i}$	$\mathbb{H}$	i, j
6	$\mathbb{HC}$	$i, j, \hat{i}$	$Ad(i)$	a	j	$\mathbb{CC}$	$i, \hat{i}$
7	$\mathbb{HC}$	$i, j, \hat{i}$	$Ad(i)\hat{\kappa}$	b	$j, \hat{i}$	$\mathcal{C}\mathbb{C}$	$i, j\hat{i}$
8	$\mathbb{HH}$	$i, j, \hat{i}, \hat{j}$	$Ad(i)$	a	j	$\mathbb{CH}$	$i, \hat{i}, \hat{j}$
9	$\mathbb{HH}$	$i, j, \hat{i}, \hat{j}$	$Ad(i)Ad(\hat{i})$	b	$j, \hat{j}$	$\mathcal{C}\mathbb{CC}$	$i, \hat{i}, j\hat{j}$
10	$\mathcal{C}\mathbb{C}$	$s, \hat{i}$	κ_p	a	s	$\mathbb{C}$	i
11	$\mathcal{C}\mathbb{C}$	s, i	κ	a	i	$\mathcal{C}$	s
12	$\mathbb{O}$	i, j, l	τ	a	l	$\mathbb{H}$	i, j
13	$\mathbb{OC}$	$i, j, l, \hat{i}$	$\hat{\kappa}$	a	$\hat{i}$	$\mathbb{O}$	i, j, l
14	$\mathbb{OC}$	$i, j, l, \hat{i}$	τ	a	l	$\mathbb{HC}$	$i, j, \hat{i}$
15	$\mathbb{OC}$	$i, j, l, \hat{i}$	$\tau\hat{\kappa}$	b	$l, \hat{i}$	$\mathcal{C}'\mathbb{H}$	$i, j, l\hat{i}$
16	$\mathbb{OH}$	$i, j, l, \hat{i}, \hat{j}$	$\mathrm{Ad}(\hat{i})$	a	$\hat{j}$	$\mathbb{OC}$	$i, j, l, \hat{i}$
17	$\mathbb{OH}$	$i, j, l, \hat{i}, \hat{j}$	τ	a	l	$\mathbb{HH}$	$i, j, \hat{i}, \hat{j}$
18	$\mathbb{OH}$	$i, j, l, \hat{i}, \hat{j}$	$\tau\,\mathrm{Ad}(\hat{i})$	b	$l, \hat{j}$	$\mathcal{C}'\mathbb{HC}$	$i, j, \hat{i}, l\hat{j}$
19	$\mathbb{OO}$	$i, j, l, \hat{i}, \hat{j}, \hat{l}$	$\hat{\tau}$	a	$\hat{l}$	$\mathbb{OH}$	$i, j, l, \hat{i}, \hat{j}$
20	$\mathbb{OO}$	$i, j, l, \hat{i}, \hat{j}, \hat{l}$	$\tau\hat{\tau}$	b	$l, \hat{l}$	$\mathcal{C}'\mathbb{HH}$	$i, j, \hat{i}, \hat{j}, l\hat{l}$

Table 1

3. Spaces of self-reflective subspaces

In Table 2 below we represent all irreducible type-I symmetric spaces as Grassmannians (including Rosenberg planes) or spaces of self-reflective subspaces in Grassmannians. For classical spaces this has been proved in [4], based on recent work of Y. Huang and N.C. Leung [7]:

Theorem 1. *All classical type-I symmetric spaces (up to coverings and local S^1 factors) are either Grassmannians $G_p(\mathbb{A}^n)$ or sets of subspaces $\{G_p(\mathbb{B}^n) \subset G_p(\mathbb{A}^n)\}$ where $\mathbb{B}$ is some self-reflective subalgebra of $\mathbb{A}$ where either $\mathbb{A} = \mathbb{K} \otimes \mathbb{L} =: \mathbb{KL}$ with $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ or $\mathbb{A}$ is a self-reflective subalgebra of $\mathbb{K} \otimes \mathbb{L}$. Some symmetric spaces allow several such descriptions, cf. Table 2.*[4]

The proof is mainly by identifying the group G of orthogonal $\mathbb{A}$-linear maps g on $\mathbb{A}^n$. E.g. if $\mathbb{A} = \mathbb{HC}$, we have on $V = \mathbb{A}^n$ two anticommuting complex structures i and j and another complex structure $\hat{i} = 1 \otimes i$ which commutes with i, j. Then $S = i\hat{i}$ with $S^2 = I$ has i-invariant (± 1)-eigenspaces $V_\pm$ which are interchanged by j. Any $g \in G$ commutes with $i, j, \hat{i}$. Thus g preserves the eigenspaces $V_\pm$ and is already determined by its restriction to V_- since $V_+ = jV_-$. Since g commutes with i, the restriction $g|V_-$ is in the unitary group of $V_- \cong \mathbb{C}^{2n}$ and hence $G = U_{2n}$.

But this proof does not apply to cases where $\mathbb{A}$ is non-associative. An algebraic reason is that $\mathbb{A}^n$ is no longer an $\mathbb{A}$-module. But there is also a geometric reason: From a representation of a symmetric space P as a space of certain subspaces in $\mathbb{A}^n$ we get an R-space structure on P, a noncompact transformation group extending the isometry group on P. We just have to replace the orthogonal $\mathbb{A}$-linear transformations by arbitrary $\mathbb{A}$-linear transformations. But most exceptional spaces are not R-spaces!

Therefore in the second part of Table 2 below we have used results of D.S.P. Leung and Chen-Nagano [8, 3] on self-reflective submanifolds. If $Q \subset P = G/K$ is a (self-)reflective subspace, then $\{Q \subset P\} = G/G_Q$, where $G_Q = \{g \in G : g(Q) = Q\}$, and G_Q contains the symmetry group of Q as a normal subgroup. Thus it is easy to identify the spaces $\{Q \subset P\}$. E.g. E_6/F_4 can be viewed as the set of all totally geodesic embeddings of $\mathbb{OP}^2$ into $\mathbb{OCP}^2$ (cf. [2], p. 313); we will write briefly $E_6/F_4 = \{\mathbb{OP}^2 \subset \mathbb{OCP}^2\}$. Here $\mathbb{A} = \mathbb{OC}$ and $\mathbb{B} = \mathbb{OR} = \mathrm{Fix}(\rho)$ with $\rho = \mathrm{id} \otimes \hat{\kappa}$ on $\mathbb{A}$ where $\hat{\kappa}$ is complex conjugation on the second tensor factor $\mathbb{C}$. But other cases, like E_6/Sp_4 and E_7/SU_8, are less obvious. It seems that those correspond to involutions ρ of type (b) on $\mathbb{A}$. In fact, the pairs $(\mathbb{B}, \mathbb{A})$ assigned to No. 15, 19 and 24 are only conjectured, following [7]. If we look for a proof, Vinberg's formula gives some hope to understand the corresponding groups, at least on the infinitesimal level. This requires an extension of Vinberg's formula to self-reflective subalgebras of $\mathbb{O} \otimes \mathbb{K}$ and also to associative algebras to allow comparison with the classical cases.

4. Conclusion

The linear algebra relevant to Riemannian symmetric spaces has two parts. Part 1 is the linear algebra over the the associative division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and it is connected to the classical symmetric spaces. Part 2 should be a restricted linear algebra over $\mathbb{O}$ which is not yet fully developed. It must be quite different. It cannot contain vectors and modules and linear maps, but projective lines and planes and even (3×3)-matrices seem to survive. We have tried a description of symmetric spaces - classical as well as exceptional - in terms of the algebras $\mathbb{A} = \mathbb{K} \otimes_{\mathbb{R}} \mathbb{L}$ with $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. For the classical spaces (avoiding $\mathbb{O}$) this is a theorem, for the exceptional ones it is just a conjecture, but with some evidence. So let's try to learn Linear Algebra 2.

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[4] The second column in Table 2 is the type of the space according to E. Cartan's classification, cf. [5]. In the last column, the space is represented either as a Grassmannian $G_p(\mathbb{A}^n)$ or as a space of subspaces $\{G_p(\mathbb{B}^n) \subset G_p(\mathbb{A}^n)\}$.

It is still incomprehensible how such an awful algebra like $\mathbb{O}\otimes\mathbb{O}$ (not associative, not alternative, no division algebra) can be related to such beautiful structures as the symmetric space E_8/SO'_{16}. It is still unclear to me why twofold tensor products of division algebras occur, but not tensor products with three or more factors. It is still unclear why some projective geometry survives the breakdown of linear algebra in the non-associative case. Elie Cartan classified symmetric spaces in 1926, but 86 years later we are still far from a full understanding of these spaces.

No.	Type	Space	dim	rk	$\mathbb{B}$	$\mathbb{A}$	Grassmannian
1	AI	U_n/O_n	$\frac{n(n+1)}{2}$	n	$\mathbb{R}$	$\mathbb{C}$	$\{\mathbb{RP}^{n-1} \subset \mathbb{CP}^{n-1}\}$
2		U_{2n}/O_{2n}	$n(2n{+}1)$	$2n$	$\mathcal{C}\mathbb{C}$	$\mathbb{HC}$	$\{G_2(\mathbb{R}^{2n}) \subset G_2(\mathbb{C}^{2n})\}$
3	AII	U_{2n}/Sp_n	$2n(n{-}1)$	n	$\mathbb{H}$	$\mathbb{HC}$	$\{\mathbb{HP}^{n-1} \subset G_2(\mathbb{C}^{2n})\}$
4	$AIII$	$U_{p+q}/(U_pU_q)$	$2pq$	p		$\mathbb{C}$	$G_p(\mathbb{C}^{p+q})$
5		$U_{2n}/(U_{2p}U_{2n-2p})$	$8p(n{-}p)$	$2p$		$\mathbb{HC}$	$G_p(\mathbb{HC}^n)$
6		$U_{2n}/(U_nU_n)$	$2n^2$	n	$\mathbb{CC}$	$\mathbb{HC}$	$\{(\mathbb{CP}^{n-1})^2 \subset G_2(\mathbb{C}^{2n})\}$
7	BDI	O_{p+q}/O_pO_q	pq	p		$\mathbb{R}$	$G_p(\mathbb{R}^{p+q})$
8		$O_{4n}/O_{4p}O_{4n-4p}$	$16p(n{-}p)$	$4p$		$\mathbb{HH}$	$G_p(\mathbb{HH}^n)$
9		O_{2n}/O_nO_n	n^2	n	$\mathcal{C}$	$\mathcal{C}\mathbb{C}$	$\{(\mathbb{RP}^{n-1})^2 \subset G_2(\mathbb{R}^{2n})\}$
10		$O_{4n}/O_{2n}O_{2n}$	$4n^2$	$2n$	$\mathcal{C}\mathbb{CC}$	$\mathbb{HH}$	$\{G_2(\mathbb{R}^{2n})^2 \subset G_4(\mathbb{R}^{4n})\}$
11	$DIII$	O_{2n}/U_n	$n(n{-}1)$	$[\frac{n}{2}]$	$\mathbb{C}$	$\mathcal{C}\mathbb{C}$	$\{\mathbb{CP}^{n-1} \subset G_2(\mathbb{R}^{2n})\}$
12		O_{4n}/U_{2n}	$2n(2n{-}1)$	n	$\mathbb{HC}$	$\mathbb{HH}$	$\{G_2(\mathbb{C}^{2n}) \subset G_4(\mathbb{R}^{4n})\}$
13	CI	Sp_n/U_n	$n(n{-}1)$	n	$\mathbb{C}$	$\mathbb{H}$	$\{\mathbb{CP}^{n-1} \subset \mathbb{HP}^{n-1}\}$
14	CII	Sp_{p+q}/Sp_pSp_q	$4pq$	p		$\mathbb{H}$	$G_p(\mathbb{H}^{p+q})$
15	EI	E_6/Sp_4	42	6	$\mathcal{C}'\mathbb{H}$?	$\mathbb{OC}$	$\{G_2(\mathbb{H}^4)/\mathbb{Z}_2 \subset \mathbb{OCP}^2\}$
16	EII	E_6/SU_6Sp_1	40	4	$\mathbb{HC}$	$\mathbb{OC}$	$\{G_2(\mathbb{C}^6) \subset \mathbb{OCP}^2\}$
17	$EIII$	$E_6/Spin_{10}U_1$	32	2		$\mathbb{OC}$	$\mathbb{OCP}^2$
18	EIV	E_6/F_4	26	2	$\mathbb{O}$	$\mathbb{OC}$	$\{\mathbb{OP}^2 \subset \mathbb{OCP}^2\}$
19	EV	E_7/SU_8	70	7	$\mathcal{C}'\mathbb{HC}$?	$\mathbb{OH}$	$\{G_4(\mathbb{C}^8)/\mathbb{Z}_2 \subset \mathbb{OHP}^2\}$
20	EVI	$E_7/SO'_{12}Sp_1$	64	4		$\mathbb{OH}$	$\mathbb{OHP}^2$
21					$\mathbb{HH}$	$\mathbb{OH}$	$\{G_4^+(\mathbb{R}^{12}) \subset \mathbb{OHP}^2\}$
22	$EVII$	E_7/E_6U_1	54	3	$\mathbb{OC}$	$\mathbb{OH}$	$\{\mathbb{OCP}^2 \subset \mathbb{OHP}^2\}$
23	$EVIII$	E_8/SO'_{16}	128	8		$\mathbb{OO}$	$\mathbb{OOP}^2$
24					$\mathcal{C}'\mathbb{HH}$?	$\mathbb{OO}$	$\{G_8^{\#}(\mathbb{R}^{16}) \subset \mathbb{OOP}^2\}$
25	EIX	E_8/E_7Sp_1	112	4	$\mathbb{OH}$	$\mathbb{OO}$	$\{\mathbb{OHP}^2 \subset \mathbb{OOP}^2\}$
26	FI	F_4/Sp_3Sp_1	28	4	$\mathbb{H}$	$\mathbb{O}$	$\{\mathbb{HP}^2 \subset \mathbb{OP}^2\}$
27	FII	$F_4/Spin_9$	16	1		$\mathbb{O}$	$\mathbb{OP}^2$
28	GI	G_2/SO_4	8	2	$\mathbb{H}$	$\mathbb{O}$	$\{\mathbb{H} \subset \mathbb{O}\}$

Table 2

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