PSEUDOHOLOMORPHIC CURVES IN $\mathbb{S}^6$ AND $\mathbb{S}^5$

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Abstract. The octonian cross product on $\mathbb{R}^7$ induces a nearly Kähler structure on $\mathbb{S}^6$, the analogue of the Kähler structure of $\mathbb{S}^2$ given by the usual (quaternionic) cross product on $\mathbb{R}^3$. Pseudoholomorphic curves with respect to this structure are the analogue of meromorphic functions. They are (super-)conformal minimal immersions. Using a slightly different method we reprove a theorem of Hashimoto [9] giving an intrinsic characterization of pseudoholomorphic curves in $\mathbb{S}^6$ and (beyond Hashimoto's work) $\mathbb{S}^5$. Instead of the Maurer-Cartan equations we use an embedding theorem into homogeneous spaces (here: $S^6 = G_2/SU_3$) involving the canonical connection. The integrability conditions must be checked only for a 3×3 matrix system instead of 7×7.

Contents

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1. Introduction

Minimal surfaces in the round 3-sphere $\mathbb{S}^3$ have an intrinsic characterization: A metric ds^2 on a simply connected Riemann surface M is the induced metric of a full conformal minimal immersion into $\mathbb{S}^3$ if and only if its Gaussian curvature K satisfies $K \leq 1$ and

$$\Delta \log(1 - K) = 4K$$

where Δ is the Laplacian of the metric ds^2.[1] The formula goes back to Ricci [11, p. 340] who actually looked at surfaces of constant mean curvature 1 in euclidean 3-space, but these are isometric to minimal surfaces in $\mathbb{S}^3$. There are similar ("Ricci-like") formulas in other situations. In $\mathbb{S}^4$, superminimal surfaces (those with trivial associated family) are characterized by the equation (cf. [7, p. 191])

$$\Delta \log(1 - K) = 6K - 2\,.$$

In the present paper, we give such characterizations for certain types of minimal surfaces in $\mathbb{S}^5$ and $\mathbb{S}^6$:

$$\Delta \log(1 - K) = 6K \tag{53}$$

for so called *pseudoholomorphic curves*[2] in $\mathbb{S}^5$ and

$$\Delta \log(1 - K) = 6K - 1 \tag{50}$$

for superminimal pseudoholomorphic curves in $\mathbb{S}^6$ (see below). General pseudoholomorphic curves in $\mathbb{S}^6$ allow a similar characterization [9] which however depends on an additional structure, a holomorphic 6-form Λ on M (which is zero in the superminimal case):

$$\Delta \log(1 - K) - (6K - 1) = |\Lambda|^2/(1 - K)^2\,. \tag{46}$$

A general theory of minimal surfaces in spheres allowing for Ricci-like characterizations was recently given in [14].

Pseudoholomorphic curves in $\mathbb{S}^6$ are the analogues of meromorphic functions on Riemann surfaces when $\mathbb{H}$ is replaced by $\mathbb{O}$. In fact, let $\mathbb{S} \in \{\mathbb{S}^2, \mathbb{S}^6\}$ be the unit sphere in the imaginary quaternions $\mathbb{H}'$ or octonions $\mathbb{O}'$, respectively. Left translation with the position vector $p \in \mathbb{S}$ induces an almost complex structure on $\mathbb{S}$ (which is integrable for $\mathbb{S} = \mathbb{S}^2$). For any Riemann surface M, a smooth mapping $f : M \to \mathbb{S}$ is *pseudoholomorphic* if its derivative $df_u : T_uM \to T_{f(u)}\mathbb{S}^6$ is complex linear with respect to this almost complex structure. For

[1] This condition makes sense even at the zeros of $1 - K$. In fact, for a minimal surface in $\mathbb{S}^3$, the expression $1 - K$ is a so called *absolute value type function* [5], the absolute value of a holomorphic function (which may have zeros) multiplied by a positive function. Then $\Delta \log(1 - K)$ is still defined at the zeros of $1 - K$.

[2] The term "curve" means complex curve, parametrized on a Riemann surface.

$\mathbb{S} = \mathbb{S}^2$ these are the meromorphic functions on M. In the present paper we are dealing with the other case $\mathbb{S} = \mathbb{S}^6$. In particular, these maps are conformal and harmonic, hence (possibly branched) minimal immersions.

The subject was started by Bryant [3] who described pseudoholomorphic curves in terms of an adapted frame, called *Frenet frame* in analogy to real curves in 3-space, and he gave examples for pseudoholomorphic curves on compact Riemann surfaces of any genus. Bolton, Vrancken and Woodword [2] characterized pseudoholomorphic curves among the minimal surfaces in $\mathbb{S}^6$. The intrinsic characterization (46) was given by Hashimoto [9]. In the present paper, we will use the same Frenet frame but our method is different from that of [3, 9]. Instead of the Maurer-Cartan equation we use an embedding theorem [6] into reductive homogeneous spaces which reduces the a-priori number of integrability conditions considerably.

In section 2 we briefly describe our method. After recalling the necessary background on octonionic computations and pseudoholomorphic maps in the 6-sphere (sections 3 - 7), we derive in section 10 the equations for the Frenet frame in terms of the canonical connection introduced in sections 8, 9. The main results are stated and proved in sections 12 for $\mathbb{S}^6$ and in 13 for $\mathbb{S}^5$ (a case which was not treated by Hashimoto [9]). We try to give complete computations with all details.

2. Embedding into a homogeneous space

Let $S = G/H$ be a Riemannian homogeneous space. If a smooth map $f : M \to G/H$ is given, there exists locally a smooth "lift" $F : M \to G$ with $f = \pi \circ F$ where $\pi : G \to G/H$ is the canonical projection. Choosing a basis $b = (b_1, \dots, b_k)$ of the tangent space $T_{p_o}S$ where $p_o = eH \in G/H$ is the base point in S, we may consider F as the "moving frame" $Fb = (F_1, \dots, F_k)$ where each $F_j = Fb_j$ is a vector field along the map f. The lift F in turn can be described by the $\mathfrak{g}$-valued one-form $\alpha = F^{-1}dF$.[3] Vice versa, if an arbitrary $\mathfrak{g}$-valued one-form α on a simply connected manifold M is given, we look for a map $F : M \to G$ with

$$dF = F\alpha \,. \tag{1}$$

This is an overdetermined system, and the local existence of solutions is equivalent to an integrability condition for the coefficient matrix α, the Maurer-Cartan equation $d\alpha = [\alpha, \alpha]$.

[3] To simplify notation, we think of G as a matrix group $G \subset \mathbb{R}^{n \times n}$.

In the present paper we replace (1) by the equation

$$\nabla F = F\beta \tag{2}$$

where ∇ is a *canonical G-connection* on $S = G/H$ (holonomy in G and parallel curvature and torsion).[4] Now β is a one-form on M which takes values in $\mathfrak{h}$ rather than in $\mathfrak{g}$. Local existence of a solution F of (2) is still equivalent to an integrability condition, but the size of the coefficient matrix β is considerably smaller. In the present case of *pseudoholomorphic curves* in $\mathbb{S}^6$, the integrability condition for β is

$$[B', B''] + (B'')_z - (B')_{\bar z} = \mathrm{diag}(\lambda,\ -\lambda/2,\ -\lambda/2) \tag{40}$$

(see section 10), where λ is the conformal factor and $\beta = B'dz + B''d\bar z$. This is due to an embedding theorem for homogeneous spaces in [6].

Our frame F along a pseudoholomorphic curve f in $\mathbb{S}^6$ is that of Bryant and Hashimoto [3, 9], up to ordering. It is an adapted frame which take care of the higher normal spaces of the immersion. In particular, F_1 is essentially the differential of f itself.

3. Octonions

A finite dimensional algebra $\mathbb{A}$ over $\mathbb{R}$ with euclidean inner product is called "normed" if $|ab| = |a||b|$ for any $a, b \in \mathbb{A}$. We have an orthogonal decomposition $\mathbb{A} = \mathbb{R}\cdot 1 \oplus \mathbb{A}'$ where $\mathbb{A}'$ is called the space of *imaginary* elements of $\mathbb{A}$. Every nonzero $a \in \mathbb{A}$ has an inverse $a^{-1} = \bar a/|a|^2$ where $\bar a = a_o - a'$ for $a = a_o + a'$ with $a_o \in \mathbb{R}$ and $a' \in \mathbb{A}'$. There are only four normed algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (real and complex numbers, quaternions and octonions), and the octonions $\mathbb{O} \cong \mathbb{R}^8$ contain all the others. Octonions are not associative, but still computations are easy if one observes the following three rules which follow almost immediately from the equation $|ab| = |a||b|$:[5]

(1) Any unit vector $a \in \mathbb{O}'$ generates a subalgebra isomorphic to $\mathbb{C}$ where a plays the rôle of i.

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[4] More precisely, a connection ∇ on TS is called *G-connection* if all its parallel displacements are given by elements of G. Then the one-form β in (2) takes values in $\mathfrak{h}$. In fact, along any curve $c : (-\epsilon, \epsilon) \to S$ starting at the base point $p_o = eH \in S$ we have $\nabla_{c'}F.v = \partial_{c'}(\tau^{-1}F).v$ for any $v \in T_{p_o}S$, where τ is the parallel transport along c. Since $\tau^{-1}F$ takes values in G and fixes p_o, it takes values in H, hence $\nabla_{c'}F \in \mathfrak{h}$.

[5] If $a \in \mathbb{O}'$ and $|a| = 1$, then $|1 \pm a| = \sqrt{2}$, hence $|1 + a||1 - a| = 2$. On the other hand, $(1 + a)\,((1 - a)x) = (1 - a)x + a(x - ax) = x - a(ax)$ for all $x \in \mathbb{O}$, and $|(1 + a)\,((1 - a)x)\,| = 2|x|$. Thus $|x - a(ax)| = 2|x|$. This is impossible unless the two vectors x and $-a(ax)$ (which have equal length) are equal, $a(ax) = -x$. This shows (1), and (2), (3) can be proved similarly.

(2) Any two orthonormal $a, b \in \mathbb{O}'$ generate a subalgebra isomorphic to $\mathbb{H}$ where a, b, ab play the rôles of i, j, k; they are associative and *anti-commutative*, $ab = -ba$.
(3) Any three orthonormal $a, b, c \in \mathbb{O}'$ with $c \perp ab$ ("normed Cayley triples") generate the algebra $\mathbb{O}$; they are *anti-associative*, $a(bc) = -(ab)c$.

Let $1, i, j, k, l, il, jl, kl$ be the standard basis of $\mathbb{O} = \mathbb{H} + \mathbb{H}l$. Then (i, j, l) is a normed Cayley triple, and so is its image $(\alpha i, \alpha j, \alpha l)$ under any automorphism α of $\mathbb{O}$; note that α is orthogonal.[6] Vice versa, given any normed Cayley triple (a, b, c), there is precisely one automorphism α of $\mathbb{O}$ with $a = \alpha i$, $b = \alpha j$, $c = \alpha l$. Thus the space of normed Cayley triples is a manifold of dimension $6 + 5 + 3 = 14$ on which the exceptional group $G_2 = \mathrm{Aut}(\mathbb{O}) \subset SO_7$ acts simply transitively. In particular, G_2 acts transitively on $\mathbb{S}^6$.

We will also need the complexified octonions $\mathbb{O}_c = \mathbb{O} \otimes \mathbb{C} = \mathbb{O} \oplus \mathsf{i}\mathbb{O}$ (we distinguish $\mathsf{i} = \sqrt{-1}$ from $i \in \mathbb{O}$). This is no longer a division algebra: there are zero devisors, e.g. $1 + \mathsf{i}a$ for any $a \in \mathbb{S}^6 \subset \mathbb{O}'$. However, analytic formulas which hold in $\mathbb{O}$ extend to $\mathbb{O}_c$. E.g. for $a \in \mathbb{O}'$ and $b \in \mathbb{O}$ we have (using rule (2))

$$a(ab) = a^2 b = -\langle a, a\rangle b\,, \tag{3}$$

and this remains true for $a \in \mathbb{O}'_c$, $b \in \mathbb{O}_c$ where $\langle\ ,\ \rangle$ is the complexified inner product. In particular $a(ab) = 0$ when $\langle a, a\rangle = 0$. Other useful formulas which extend for all $a, b, c \in \mathbb{O}_c$ are

$$\langle ab, ac\rangle = \langle a, a\rangle\langle b, c\rangle \tag{4}$$

and the antisymmetry of $\langle ab, c\rangle$ in all three variables.

As $\mathbb{O}$ is decomposed into planes that are invariant under left multiplication with $\mathbb{C} \subset \mathbb{O}$, we may decompose $\mathbb{O}_c$ into free $\mathbb{C}_c$-modules where $\mathbb{C}_c = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of $\mathbb{C}$. A *complex Cayley triple* is a triple (a, b, c) in $\mathbb{O}'_c$ where a lies in $\mathbb{C}_c$ (or in an isomorphic subalgebra) and where b, c belong to two perpendicular $\mathbb{C}_c$-modules. Like its real analogue, a complex Cayley triple is anti-associative, $(ab)c = -a(bc)$.

4. The nearly Kähler structure on $\mathbb{S}^6$

The 6-sphere $\mathbb{S}^6$ plays a similar rôle for the octonions $\mathbb{O}$ as the 2-sphere $\mathbb{S}^2$ for the quaternions $\mathbb{H}$: they are unit spheres in $\mathbb{A}'$, the imaginary part of the division algebra $\mathbb{A} = \mathbb{O}, \mathbb{H}$, respectively, a fact which

[6]Any automorphism of $\mathbb{O}$ is orthogonal: it preserves real and imaginary octonions since real octonions are real multiples of 1 and imaginary octonions are those which square to negative real multiples of 1. Thus an automorphism preserves the conjugation $a^* = \mathrm{Re}\ a - \mathrm{Im}\ a$ and also the norm $|a|^2 = a^*a$ for any $a \in \mathbb{O}$.

has been observed and applied in [8]. Each $p \in \mathbb{S}$ satisfies $(L_p)^2 = -I$ where $L_p : x \mapsto px$ denotes the left multiplication with p. Hence L_p is a complex structure preserving the plane Span $\{1, p\}$ and its orthogonal complement, the tangent space $T_p\mathbb{S}$. Thus $J_p := L_p|T_p\mathbb{S}$ is a complex structure on $T_p\mathbb{S}$ and defines an almost complex structure J on $\mathbb{S}$. It is convenient to use the *cross product* $a \times b$ which is the imaginary ($\mathbb{A}'$-) part of the product ab for any $a, b \in \mathbb{A}'$:

$$a \times b = (ab)' = \left\{ \begin{matrix} ab & \text{when} & a \perp b, & \\ 0 & \text{when} & a, b & \text{lin. dependent} \end{matrix} \right\} .$$

Then each J_p extends to a linear map on $\mathbb{A}'$,

$$J_p(v) = p \times v \,, \tag{5}$$

and the derivative of the matrix-valued linear map $J : \mathbb{A}' \to \mathrm{End}(\mathbb{A}') : p \mapsto J_p$ is $(\partial_v J)w = v \times w$. Denoting by $D = \partial^T$ the Levi-Civita derivative on $\mathbb{S}$, we have

$$(D_v J)w = (v \times w)_{p^\perp} = v \times w - \langle v \times w, p\rangle\, p \tag{6}$$

where $p \in \mathbb{S}$ is the position vector and $v, w \in T_p\mathbb{S} = p^\perp$. In particular $(\partial_v J)v = v \times v = 0$ and therefore

$$(D_v J)v = 0. \tag{7}$$

A Riemannian manifold with an almost complex structure J with this property is called *nearly Kähler*.[7]

An orthogonal linear map g on $\mathbb{O}'$ which preserves the almost complex structure J satisfies $gJ_p(v) = J_{gp}(gv)$ for any $p, v \in \mathbb{O}'$ with $v \perp p$. By (5) this is equivalent to $g(pv) = (gp)(gv)$ which holds if and only if $g \in G_2 = \mathrm{Aut}(\mathbb{O}) \subset SO_7$. Thus G_2 is precisely the group of isometries g on $\mathbb{S}^6$ which are *pseudoholomorphic*, that is their differentials $dg_p : T_a\mathbb{S}^6 \to T_{gp}\mathbb{S}^6$ are complex linear with respect to the complex structures given by J on the tangent spaces of $\mathbb{S}^6$. The stabilizer subgroup $H = (G_2)_p$ of any $p \in \mathbb{S}^6$ (say: $p = l$) preserves the tangent space $T_p\mathbb{S}^6$ and its complex structure J_p, making $T_p\mathbb{S}^3$ a 3-dimensional complex vector space. Identifying $(T_p\mathbb{S}^3, J_p)$ with $\mathbb{C}^3$ we obtain $H \subset U_3$. But H preserves also the antisymmetric 3-form $\langle uv, w\rangle$ on $T_p\mathbb{S}^6$ which can be viewed as the real part of a complex determinant, thus $H \subset SU_3$, and by dimension reasons we have equality $H = SU_3$.

[7] In the case of $\mathbb{S}^2$ we even obtain $DJ = 0$ (Kähler property) since $v \times w$ is normal when v, w are tangent vectors, hence $(D_v J)w = (v \times w)^T = 0$.

5. Pseudoholomorphic curves

Let M be a Riemann surface. A smooth map $f : M \to \mathbb{S}^6$ is called *pseudoholomorphic* if it is holomorphic with respect to this almost complex structure $J_p v = p \times v$. In other words, if $z = x + iy$ is a conformal coordinate on M, the corresponding partial derivatives f_x, f_y satisfy

$$f \times f_x = f_y\,, \quad f \times f_y = -f_x\,. \tag{8}$$

Clearly, such map is conformal since $|f_x| = |f_y|$ and $f_x \perp f_y$. Further f is *harmonic*, that is $f_{xx} + f_{yy}$ is a normal vector, a multiple of f. In fact, differentiating (8) we obtain

$$\begin{array}{rclcl} f_{yy} & = & (f \times f_x)_y & = & f_y \times f_x + f \times f_{xy}\,, \\ f_{xx} & = & -(f \times f_y)_x & = & -f_x \times f_y - f \times f_{yx}\,, \end{array} \tag{9}$$

and hence

$$f_{yy} + f_{xx} = 2 f_y \times f_x\,, \tag{10}$$

$$f_{yy} - f_{xx} = 2 f \times f_{xy}\,. \tag{11}$$

The first equation (10) shows that f is harmonic: $f_y \times f_x$ is proportional to f since by (8), f, f_x, f_y span a quaternion subalgebra. Moreover

$$f_{yx} = (f \times f_x)_x = f \times f_{xx} = J f_{xx}. \tag{12}$$

It is convenient to use the complex derivatives $f_z = \frac{1}{2}(f_x - \mathsf{i} f_y)$ and $f_{zz} = \frac{1}{4}\left((f_x - \mathsf{i} f_y)_x - \mathsf{i}(f_x - \mathsf{i} f_y)_y\right) = \frac{1}{4}\left(f_{xx} - f_{yy} - 2\mathsf{i} f_{xy}\right) \overset{(11)}{=} -\frac{1}{2}(J + \mathsf{i}) f_{xy}$. Hence

$$\begin{array}{rcl} f_z & = & -(J + \mathsf{i}) f_y / 2\,, \\ f_{zz} & = & -(J + \mathsf{i}) f_{xy} / 2\,. \end{array} \tag{13}$$

Since $(J - \mathsf{i})(J + \mathsf{i}) = 0$, these vectors belong to the i-eigenspace E_+ of $J_f : v \mapsto f \times v$ on $T_f \mathbb{S}^6$. This is an *isotropic* subspace, i.e. $\langle v, v \rangle = 0$ for all $v \in E_+$: If $v = (J + \mathsf{i}) a$, then $\langle v, v \rangle = \langle Ja, Ja \rangle - \langle a, a \rangle + 2\mathsf{i} \langle Ja, a \rangle = 0$.

Lemma 5.1. *Putting $\lambda = \langle f_z, f_{\bar z} \rangle = |f_z|^2$ and $l = \log \lambda$, we have*

$$\begin{array}{rcl} f_{zz} & = & f_{zz}^{\perp} + l_z f_z\,, \\ (f_z)_{\bar z} & = & -\lambda f\,, \\ (f_{zz}^{\perp})_{\bar z} & = & -(\lambda + l_{z \bar z}) f_z\,. \end{array} \tag{14}$$

Proof. To prove the first equation we note $\langle f_{zz}, f_z \rangle = \frac{1}{2} \langle f_z, f_z \rangle_z = 0$ and $\langle f_{zz}, f_{\bar z} \rangle = \lambda_z - \langle f_z, f_{z \bar z} \rangle = \lambda_z$ since $\langle f_z, f_{z \bar z} \rangle = \frac{1}{2} \langle f_z, f_z \rangle_{\bar z} = 0$. Hence $f_{zz} - f_{zz}^{\perp} = f_{zz}^T = \frac{1}{\lambda} \langle f_{zz}, f_{\bar z} \rangle f_z = (\lambda_z / \lambda) f_z = l_z f_z$.

The second equation follows since $4 f_{z \bar z} = (f_x - \mathsf{i} f_y)_x + \mathsf{i}(f_x - \mathsf{i} f_y)_y = f_{xx} + f_{yy}$, and this is a multiple of f. To determine the multiple we we compute the inner product $\langle f_{z\bar z}, f\rangle = \langle f_z, f\rangle_{\bar z} - \langle f_z, f_{\bar z}\rangle = -\lambda$ since $\langle f_z, f\rangle = \frac{1}{2}\langle f, f\rangle_z = 0$. This shows the second equality.

The third equality follows from the two previous ones: From $f_{zz}^{\perp} = f_{zz} - l_z f_z$ we have $(f_{zz}^{\perp})_{\bar z} = f_{zz\bar z} - (l_z f_z)_{\bar z} = -(\lambda f)_z - l_{z\bar z} f_z + l_z \lambda f = -(\lambda + l_{z\bar z}) f_z$, using $\lambda_z = l_z \lambda$. □

As a consequence, f_z and $f_{zz}^{\perp}$ are holomorphic sections of the complexified tangent and normal bundles T^c and N^c of $f : M \to \mathbb{S}$, since $(f_z)_{\bar z}$ and $(f_{zz}^{\perp})_{\bar z}$ have zero projection to T^c and N^c, respectively. Thus the isotropic subbundles $T' = \mathbb{C} f_z$ and $N_1' = \mathbb{C} f_{zz}^{\perp}$ are well determined even at possible zeros of these sections, and by isotropy the same holds for the real bundles T and N_1, the tangent bundle and the first normal bundle of f. Hence along f, the tangent bundle of $\mathbb{S}^6$ splits into three J-invariant orthogonal plane bundles, $f^*(T\mathbb{S}^6) = T \oplus N_1 \oplus N_2$.

The full $(+\mathrm{i})$-eigenspace $E_+ = T_f'\mathbb{S}^6$ is spanned by

$$
\begin{aligned}
F_1 &= f_z\,, \\
F_2 &= f_{zz}^{\perp}\,, \\
F_3 &= \overline{F_1 F_2} \;=\; f_{\bar z} \times f_{\bar z\bar z}\,.
\end{aligned}
\tag{15}
$$

The third line follows since (f, F_1, F_2) is a complex Cayley triple, hence $f(F_1F_2) = -(fF_1)F_2 = -\mathrm{i}F_1F_2$ and therefore $\overline{F_1F_2} \in E_+$. In analogy to the theory of curves in euclidian space $\mathbb{R}^3$, we will call $F = (F_1, F_2, F_3)$ the *Frenet frame* of f, as was suggested in [3].

The three vectors F_1, F_2, F_3 together with their complex conjugates $\bar F_1, \bar F_2, \bar F_3$ form bases of the complexified bundles T^c, N_1^c, N_2^c, respectively, and the only nonzero inner products are

$$
\langle F_1, \bar F_1\rangle =: \lambda\,, \quad \langle F_2, \bar F_2\rangle =: \mu\,, \quad \langle F_3, \bar F_3\rangle = 2\lambda\mu. \tag{16}
$$

The last equality is seen as follows: If $F_1 = (f + \mathrm{i})a$ and $F_2 = (f + \mathrm{i})b$, then $F_1F_2 = (fa + \mathrm{i}a)(fb + \mathrm{i}b) = (fa)(fb) - ab + \mathrm{i}\,((fa)b + a(fb))$. If (f, a, b) is an (unnormed) Cayley triple, then so is (f, fa, b), and $(fa)(fb) = -((fa)f)b = -ab$ (using $|f| = 1$) while $a(fb) = -(af)b = (fa)b$. Thus $F_1F_2 = -2ab + 2\mathrm{i}(fa)b$, and $|F_1F_2|^2 = 8|a|^2|b|^2$ while $|F_1|^2|F_2|^2 = 4|a|^2|b|^2$.

Remark 1. Later we will also use the *normalized Frenet frame*

$$
F_1^o = F_1/\sqrt{\lambda}, \quad F_2^o = F_2/\sqrt{\mu}, \quad F_3^o = F_3/\sqrt{2\lambda\mu}. \tag{17}
$$

Corollary 5.1. *Let $f : M \to \mathbb{S}^6$ pseudoholomorphic and z a conformal coordinate on M. Then $\mu = |f_{zz}^{\perp}|^2$ depends on $\lambda = |f_z|^2$:*

$$
\mu = \lambda^2(1 - K) = \lambda(\lambda + l_{z\bar z}) \quad \textit{where} \quad l = \log\lambda. \tag{18}
$$

Proof. From $\langle f_{zz}^\perp, f_{\bar z}\rangle = 0$ we obtain using the third equation of (14):

$$0 = \langle f_{zz}^\perp, f_{\bar z}\rangle_{\bar z} = -(\lambda + l_{z\bar z})\langle f_z, f_{\bar z}\rangle + \langle f_{zz}^\perp, f_{\bar z\bar z}\rangle = -(\lambda + l_{z\bar z})\lambda + \mu\,.$$

The first equality in (18) follows since the Gaussian curvature K of the induced metric $ds^2 = 2\lambda \cdot dz\, d\bar z$ on M satisfies

$$\lambda K = -(\log\lambda)_{z\bar z} = -l_{z\bar z},$$

thus $\lambda(1-K) = \lambda + l_{z\bar z}$. □

Remark 2. Equation (18) is just the *Gauss equation* (G) for the conformal minimal immersion $f : M \to \mathbb{S}^6$:

$$4\lambda^2(K-1) = |f_x|^2|f_y|^2(K-1) \overset{(G)}{=} \langle f_{xx}^\perp, f_{yy}^\perp\rangle - |f_{xy}^\perp|^2 \overset{*}{=} -2|f_{xx}^\perp|^2 = -4\mu.$$

For "$\overset{*}{=}$" recall that $f_{yy}^\perp = -f_{xx}^\perp$ (harmonicity) and $f_{xy} = Jf_{xx}$ (see (12). Further we have used (13) to see

$$\begin{array}{rcrcrcl} 2\lambda & = & 2|f_z|^2 & = & |f_x|^2 & = & |f_y|^2, \\ 2\mu & = & 2|f_{zz}^\perp|^2 & = & |f_{xx}^\perp|^2 & = & |f_{xy}^\perp|^2. \end{array}$$

6. The generalized Hopf differentials

For any conformal harmonic map $f : M \to \mathbb{S}^n$ on a Riemann surface M one considers the higher fundamental forms

$$A_k(v_1, \dots, v_k) = (\partial_{v_1} \dots \partial_{v_k} f)^{N_{k-1}}$$

for arbitrary tangent vectors $v_1, \dots, v_k$, where $N_0 = T$ is the tangent space and N_{k-1} for $k \ge 2$ the $(k-1)$-th normal space[8] of the surface f, and $(\ )^{N_{k-1}}$ denotes the orthogonal projection into this space. Using a conformal coordinate z on M, the harmonicity of f yields the vanishing of all mixed components of A_k (those involving both dz and $d\bar z$). Thus

$$A_k = B_k + \overline{B}_k \quad \text{with} \quad B_k = \left(\left(\frac{\partial}{\partial z}\right)^k f\right)^{N_{k-1}} dz^k\,,$$

see [12] for details. The *generalized Hopf differential* is the symmetric $2k$-form on M defined by

$$\Lambda_k = \langle B_k, B_k\rangle\,.$$

The first Hopf differential $\Lambda_1 = \langle f_z, f_z\rangle\, dz^2$ vanishes by conformality of f, and the second one $\Lambda_2 = \langle f_{zz}^\perp, f_{zz}^\perp\rangle\, dz^4$ is the classical Hopf differential which is holomorphic for every conformal harmonic map. More generally, Λ_k is holomorphic if $\Lambda_1, \dots, \Lambda_{k-1}$ vanish everywhere, cf. [12].

[8]Putting E_k the span of all derivatives of f with degree up to k where $k \ge 2$, we define N_{k-1} recursively as the orthogonal complement of N_{k-2} in E_k, where N_0 is the tangent space, the span of the first derivatives.

If M is compact of genus 0, all holomorphic differentials vanish, hence all Λ_k are zero. This is the *superminimal* case investigated first by Calabi [4].

In our case of pseudoholomorphic maps $f : M \to \mathbb{S}^6$, we have always $\Lambda_2 = 0$ since $f_{zz}^\perp$ lies in the isotropic space E_+. Therefore $\Lambda_3 = \langle f_{zzz}^{N_2}, f_{zzz}^{N_2}\rangle\, dz^6$ is holomorphic.[9] For completeness and to introduce notation we give a direct proof.

Lemma 6.1. *Let $f : M \to \mathbb{S}^6$ a pseudoholomorphic curve and z a conformal coordinate on M. Then the function $h := \langle f_{zzz}, f_{zzz}\rangle$ is holomorphic with*

$$h = \langle f_{zzz}^{N_2}, f_{zzz}^{N_2}\rangle = \langle (F_2)_z, (F_2)_z\rangle = \left\langle (F_2)_z^{N_2}, (F_2)_z^{N_2}\right\rangle \,, \tag{19}$$

and $\Lambda_3 = h(z)dz^6$.

Proof. $\langle f_{zzz}, f_{zzz}\rangle_{\bar z} = 2\langle f_{zzz\bar z}, f_{zzz}\rangle = -2\langle (\lambda f)_{zz}, f_{zzz}\rangle = 0$ since f_{zzz} is perpendicular to f, f_z, f_{zz}. In fact, $\langle f, f_{zzz}\rangle = \langle f, f_{zz}\rangle_z = 0$ since $\langle f, f_{zz}\rangle = -\langle f_z, f_z\rangle = 0$, further $\langle f_z, f_{zzz}\rangle = -\langle f_{zz}, f_{zz}\rangle = 0$ and $\langle f_{zz}, f_{zzz}\rangle = \frac{1}{2}\langle f_{zz}, f_{zz}\rangle_z = 0$. Thus h is holomorphic and $h(z)dz^6$ defines a holomorphic 6-form on M.

From (14) we have $f_{zz} = F_2 + l_z f_z$, and thus $(f_{zz} - F_2)_z = (l_z f_z)_z$ belongs to the span of f_z and f_{zz} which is part of the isotropic subspace E_+. Further, since $f_{zzz} \perp f_z, f_{zz}$, we have $f_{zzz} - f_{zzz}^{N_2} \in \mathrm{Span}\,(f_z, f_{zz})$. (The components of f_{zzz} proportional to $f_{\bar z}, f_{\bar z\bar z}$ involve the inner products with f_z, f_{zz} which are zero.) Thus $h = \langle f_{zzz}, f_{zzz}\rangle = \langle (F_2)_z, (F_2)_z\rangle$ $= \langle f_{zzz}^{N_2}, f_{zzz}^{N_2}\rangle$, and $h(z)dz^6 = \Lambda_3$. Moreover, $(F_2)_z \perp f, F_1, F_2$, hence $(F_2)_z - (F_2)_z^{N_2} \in \mathrm{Span}\,\{F_1, F_2\}$, and this component does not contribute to the inner product $\langle (F_2)_z, (F_2)_z\rangle$. This proves the last equality in (19). □

7. The derivatives of the Frenet frame

Proposition 7.1. *Let $f : M \to \mathbb{S}^6$ be a pseudoholomorphic curve with Frenet frame F_1, F_2, F_3 as in (15), corresponding to a conformal coordinate z on M. Let $\lambda = |F_1|^2$, $\mu = |F_2|^2$ and $l = \log\lambda$, $m = \log\mu$. Then:*

[9] A conformal harmonic map $f : M \to \mathbb{S}^{2m}$ with all $\Lambda_k = 0$ but the highest one Λ_{m-1} (which then must be holomorphic) is called *superconformal*.

$$
\begin{array}{rlllll}
(F_1)_z = & l_z F_1 & + F_2\,, & & & \\
(F_2)_z = & & m_z F_2 & + (\mathrm{i}h/2\lambda\mu) F_3 & - (\mathrm{i}/2)\bar{F}_3\,, \\
(F_3)_z = & & & & \mathrm{i}\lambda \bar{F}_2\,, \\
(F_1)_{\bar{z}} = & & & & -\lambda f\,, \\
(F_2)_{\bar{z}} = & -\frac{\mu}{\lambda} F_1\,, & & & \\
(F_3)_{\bar{z}} = & & (\mathrm{i}h/\mu)F_2 & + (l_{\bar{z}} + m_{\bar{z}})F_3\,. &
\end{array} \tag{20}
$$

Proof. The equations for $(F_1)_z$, $(F_1)_{\bar{z}}$ and $(F_2)_{\bar{z}}$ follow directly from (14) using $\lambda + l_{z/z} = \mu/\lambda$, see (18). The equation for $(F_3)_z = \overline{(f_z \times f_{zz}^{\perp})_{\bar{z}}}$ is proved as follows:

$$
\begin{array}{rcl}
(f_z \times f_{zz}^{\perp})_{\bar{z}} & = & f_{z\bar{z}} \times f_{zz}^{\perp} + f_z \times (f_{zz}^{\perp})_{\bar{z}} \\
& \overset{(14)}{=} & -\lambda f \times f_{zz}^{\perp} - \dfrac{\mu}{\lambda} f_z \times f_z \\
& = & -\mathrm{i}\lambda f_{zz}^{\perp}\,.
\end{array}
$$

The equations for $(F_2)_z$ and $(F_3)_{\bar{z}}$ are proved in the subsequent two lemmas. ∎

Lemma 7.1.

$$(f_{zz}^{\perp})_z = m_z f_{zz}^{\perp} + \mathrm{i}h/(2\lambda\mu) f_{\bar{z}} \times f_{\bar{z}\bar{z}} - (\mathrm{i}/2) f_z \times f_{zz} \tag{21}$$

where $l = \log\lambda$ *and* $m = \log\mu$.

Proof.

$$
\begin{array}{rclcll}
\langle (f_{zz}^{\perp})_z, f_z\rangle & = & -\langle f_{zz}^{\perp}, f_{zz}\rangle & = & 0\,, & (a) \\
\langle (f_{zz}^{\perp})_z, f_{\bar{z}}\rangle & = & -\langle f_{zz}^{\perp}, f_{\bar{z}z}\rangle \;=\; \langle f_{zz}^{\perp}, \lambda f\rangle & = & 0\,, & (b) \\
\langle (f_{zz}^{\perp})_z, f_{zz}^{\perp}\rangle & = & (1/2)\langle f_{zz}^{\perp}, f_{zz}^{\perp}\rangle_z & = & 0\,, & (c) \\
\langle (f_{zz}^{\perp})_z, f_{\bar{z}\bar{z}}^{\perp}\rangle & = & \langle f_{zz}^{\perp}, f_{\bar{z}\bar{z}}^{\perp}\rangle_z + \langle f_{zz}^{\perp}, (\lambda f)_{\bar{z}}\rangle & = & \mu_z\,, & (d) \\
\langle (f_{zz}^{\perp})_z, f_{\bar{z}} \times f_{\bar{z}\bar{z}}^{\perp}\rangle & = & \langle f_{zz}^{\perp}, \lambda f \times f_{\bar{z}\bar{z}}^{\perp} + f_{\bar{z}} \times (\lambda f)_{\bar{z}}\rangle & = & -\mathrm{i}\lambda\mu\,. & (e)
\end{array}
$$

The last equation (e) tells us

$$\langle (F_2)_z, F_3\rangle = -\mathrm{i}\lambda\mu. \tag{e}$$

It remains to compute $\langle (F_2)_z, \bar{F}_3\rangle$, using

$$h = \langle (F_2)_z^{N_2}, (F_2)_z^{N_2}\rangle.$$

We have

$$2\lambda\mu (F_2)_z^{N_2} = \langle (F_2)_z, \bar{F}_3\rangle F_3 + \langle (F_2)_z, F_3\rangle \bar{F}_3$$

and hence

$$
\begin{array}{rcl}
(2\lambda\mu)^2 h & = & 2\,\langle (F_2)_z, \bar{F}_3\rangle \cdot \langle (F_2)_z, F_3\rangle \cdot \langle F_3, \bar{F}_3\rangle \\
& = & 2\,\langle (F_2)_z, \bar{F}_3\rangle \cdot (-\mathrm{i}\lambda\mu) \cdot 2\lambda\mu
\end{array}
$$

from which we find the missing equation:

$$\langle (F_2)_z, \bar F_3\rangle = \mathsf{i}h. \tag{f}$$

From $(a), (b), (c)$ we see[10] that $(F_2)_z = aF_2 + bF_3 + c\bar F_3$, and further

$$\begin{array}{rcccl} a\,\langle F_2, \bar F_2\rangle &=& \langle (F_2)_z, \bar F_2\rangle & \overset{(d)}{=} & \mu_z\,, \\ b\,\langle F_3, \bar F_3\rangle &=& \langle (F_2)_z, \bar F_3\rangle & \overset{(f)}{=} & \mathsf{i}h\,, \\ c\,\langle \bar F_3, F_3\rangle &=& \langle (F_2)_z, F_3\rangle & \overset{(e)}{=} & -\mathsf{i}\lambda\mu\,. \end{array}$$

Thus

$$\begin{array}{rcccc} a &=& \mu_z/\mu &=& m_z\,, \\ b &=& && \mathsf{i}h/(2\lambda\mu)\,, \\ c &=& -\mathsf{i}\lambda\mu/(2\lambda\mu) &=& -\mathsf{i}/2\,. \end{array} \qquad \square$$

Lemma 7.2.

$$(f_z \times f_{zz})_z = -(\mathsf{i}h/\mu) f^\perp_{\bar z\bar z} + (l_z + m_z) f_z \times f_{zz}. \tag{22}$$

Proof. We compute the components of $(f_z \times f_{zz})_z$. Using $f_z \times f_{zz} \in N_2^c \perp T^c \oplus N_1^c$, we obtain:

$$\begin{array}{rcccc} \langle (f_z\times f_{zz})_z, f_z\rangle &=& -\langle f_z\times f_{zz}, f_{zz}\rangle &=& 0\,, \\ \langle (f_z\times f_{zz})_z, f_{\bar z}\rangle &=& \langle f_z\times f_{zz}, \lambda f\rangle &=& 0\,, \\ \langle (f_z\times f_{zz})_z, f^\perp_{zz}\rangle &=& -\langle f_z\times f_{zz}, (f^\perp_{zz})_z\rangle &\overset{(21)}{=}& -\mathsf{i}h\,, \\ \langle (f_z\times f_{zz})_z, f_{\bar z\bar z}\rangle &=& \langle f_z\times f_{zz}, (\lambda f)_{\bar z}\rangle &=& 0\,, \\ \langle (f_z\times f_{zz})_z, f_z\times f_{zz}\rangle &=& \langle f_z\times f_{zz}, f_z\times f_{zz}\rangle_z/2 &=& 0\,, \\ \langle (f_z\times f_{zz})_z, f_{\bar z}\times f_{\bar z\bar z}\rangle &\overset{*}{=}& \langle f_z\times f_{zz}, f_{\bar z}\times f_{\bar z\bar z}\rangle_z &=& 2(\lambda\mu)_z \end{array}$$

where "$\overset{*}{=}$" follows since $(f_{\bar z} \times f^\perp_{\bar z\bar z})_z = \mathsf{i}\lambda f^\perp_{\bar z\bar z} \perp N_2$. Thus we obtain $(f_z \times f_{zz})_z = a f^\perp_{\bar z\bar z} + b f_z \times f_{zz}$ with

$$\begin{array}{rcccc} a\cdot\mu &=& \langle (f_z\times f_{zz})_z, f^\perp_{zz}\rangle &=& -\mathsf{i}h\,, \\ b\cdot 2\lambda\mu &=& \langle (f_z\times f_{zz})_z, f_{\bar z}\times f_{\bar z\bar z}\rangle &=& 2(\lambda\mu)_z \end{array}$$

which shows $a = -\mathsf{i}h/\mu$ and $b = \log(\lambda\mu)_z = l_z + m_z$. $\square$

[10] Recall that by (16) any $v \in T_f\mathbb{S} = f^\perp$ has the representation $v = w + \bar w$ with

$$w = \langle v, \bar F_1\rangle F_1/\lambda + \langle v, \bar F_2\rangle F_2/\mu + \langle v, \bar F_3\rangle F_3/(2\lambda\mu)\,.$$

8. The canonical G_2 connection

The three vectors $F_1 = f_z$, $F_2 = f_{zz}^\perp$, $F_3 = f_{\bar z} \times f_{\bar z\bar z}$ defined in (15) (spanning the isotropic subspace $E_+ = \{v \in \mathbb{O}'_c : f \times v = \mathrm{i}v\}$) are positive real multiples of $i - \mathrm{i}li$, $j - \mathrm{i}lj$, $k - \mathrm{i}lk$, up to transformation with some element of $G_2 = \mathrm{Aut}(\mathbb{O})$. Thus, up to positive factors, $F = (F_1, F_2, F_3)$ can be considered as a moving G_2-frame, a section of the SU_3-principal bundle $G_2 \to G_2/SU_3 = \mathbb{S}^6$, pulled back to M via f. But as we see from Proposition 7.1, the derivative DF cannot be expressed in terms of F alone; one also needs $\bar F$. The reason is that covariant derivatives on $\mathbb{S}^6$ relies on the Levi-Civita parallel displacements which unfortunately does not preserve J, it is not in G_2. Therefore we will use another connection ∇ on $\mathbb{S}^6$, whose parallel displacements belong to G_2, a *G_2-connection* or *hermitian connection*. Thus we will derive formulas of the type $\nabla' F = FB'$ and $\nabla'' F = FB''$ for some complex 3×3-matrices B', B''. It turns out that B', B'' depend only on the metric coefficients of the surface f and some given holomorphic 6-form Λ_3; this will prove existence and uniqueness of pseudoholomorphic maps.

A G_2-connection $\nabla = D + A$ needs to make J parallel,

$$0 = \nabla_v J = D_v J + [A_v, J]$$

where $(D_v J)w = (v \times w)_{p^\perp}$ for $v, w \in T_p\mathbb{S}^6 = p^\perp$. Thus $[A_v, J] = -D_v J$. We may split $A_v = A_v^+ + A_v^-$ where A_v^+ commutes with J and A_v^- anticommutes with J. Then $-D_v J = [A_v, J] = [A_v^-, J] = 2A_v^- J$, hence $A_v^- = \frac{1}{2}(D_v J)J$ while A_v^+ is unrestricted.

Among the G_2-connections there is the *canonical connection* (see also [1]) which has the additional property that G_2 acts on $\mathbb{S}^6$ by affine transformations: $\nabla_{gV}(gW) = g(\nabla_V W)$ for any $g \in G_2$ and any two tangent vector fields V, W on $\mathbb{S}^6$. Clearly $G_2 \subset SO_7$ is affine also for the Levi-Civita connection D, hence it keeps $A = \nabla - D$ invariant. In particular, fixing a base point $p \in \mathbb{S}^6$, say $p = l$, the tensor A at p is invariant under the isotropy group SU_3 at l. Thus the map

$$v \mapsto A_v^+ : T_p\mathbb{S}^6 = \mathbb{C}^3 \to \mathbb{C}^{3\times 3}$$

is SU_3-equivariant. The group SU_3 acts on the matrix space $\mathbb{C}^{3\times 3}$ by conjugation, splitting it into two equivalent subrepresentations (hermitian and antihermitian matrices) both of which are irreducible up to a one-dimensional fixed space. Thus there is no nonzero equivariant linear map $\mathbb{C}^3 \to \mathbb{C}^{3\times 3}$. Therefore the canonical connection satisfies $A_v^+ = 0$, hence $A_v = A_v^- = \frac{1}{2}(D_v J)J$ and therefore

$$\nabla_v = D_v + A_v, \quad 2A_v = (D_v J)J. \tag{23}$$

Now $\nabla_v J = [\nabla_v, J] = [D_v, J] + [A_v, J] = 0$.

9. Canonical torsion and curvature on $\mathbb{S}^6$

It is well known that a canonical connection has parallel torsion and curvature tensors which we are going to compute now. Let us put

$$S_v = D_v J\,. \tag{24}$$

Since $J_p v = p \times v$ for any $p \in \mathbb{S}^6$ and $v \in T_p\mathbb{S}^6 = p^\perp$, we have $S_v w = (D_v J)w = (v \times w)_{p^\perp} = v \times w - \langle p, v \times w\rangle p$, and since $\langle p, v\times w\rangle = \langle p \times v, w\rangle = \langle Jv, w\rangle$, we obtain

$$S_v w = v \times w - \langle Jv, w\rangle p = (vw)^{T_p\mathbb{S}} \tag{25}$$

where p is the position vector, $v, w \in T_p\mathbb{S}^6$ and $(\ )^{T_p\mathbb{S}}$ denotes the projection onto $T_p\mathbb{S}$. Using the fact that the parallel displacements of ∇ belong to the group G_2 which preserves the cross product and the inner product, it is clear that S is a ∇-parallel tensor, see [1, Lemma 2.4] for a direct proof. Note that $2A = SJ = -JS$ since $0 = D(J^2) = SJ + JS$. Further, $S_v w = -S_w v$ by (7).

The torsion tensor of ∇ is

$$T(v,w) = \nabla_v w - \nabla_w v - [v,w] = A_v w - A_w v\,.$$

We have $2A_v w = S_v J w = -JS_v w$, and thus $A_v w = -A_w v$. Hence

$$T(v,w) = S_v J w = -JS_v w\,, \tag{26}$$

which shows again that T is ∇-parallel since so are S and J.

We want to compute S in terms our frame $(F, \bar F)$. By (17), (13), and (15), F_j is a real multiple of

$$F_j^o = (e_j - \mathsf{i} f e_j)/\sqrt{2}$$

where $e_1, e_2, e_3 \in \mathbb{O}'$ is an orthonormal 3-frame perpendicular to f with $e_3 = e_1 e_2$. Since

$$\begin{aligned}
(e_i - \mathsf{i} f e_i)(e_j - \mathsf{i} f e_j) &= 2(e_k + \mathsf{i} f e_k)\,,\\
(e_i - \mathsf{i} f e_i)(e_j + \mathsf{i} f e_j) &= 0\,,\\
(e_i - \mathsf{i} f e_i)(e_i + \mathsf{i} f e_i) &= -2 + 2\mathsf{i} f\,.
\end{aligned}$$

for $(i,j,k) = (1,2,3)$ up to cyclic permutations, we have from (25)

$$S_{F_i^o} F_j^o = \sqrt{2}\bar F_k^o, \quad S_{F_i^o} \bar F_j^o = 0, \quad S_{F_i^o}\bar F_i^o = 0\,. \tag{27}$$

The real factors are given by (17). Thus

Lemma 9.1.

$$\begin{aligned}
S_{F_1} F_2 &= \bar F_3\,,\\
S_{F_2} F_3 &= 2\,\mu \bar F_1\,,\\
S_{F_3} F_1 &= 2\,\lambda \bar F_2\,,\\
S_{F_j} \bar F_k &= 0 \qquad \forall j,k\,.
\end{aligned} \tag{28}$$

$\square$

Recalling $2A = SJ$ and $JF_j = \mathrm{i}F_j$, we obtain:

Corollary 9.1. *For $A' = A_{F_1}$ and $A'' = A_{\bar F_1}$ we have*

$$\begin{array}{rcl} 2A'F_1 &=& 0, \\ 2A'F_2 &=& \mathrm{i}\bar F_3, \\ 2A'F_3 &=& -2\mathrm{i}\lambda \bar F_2, \\ 2A''F_1 &=& 0 \\ 2A''F_2 &=& 0, \\ 2A''F_3 &=& 0. \end{array}$$

$\square$

Next we compute the curvature tensor R of ∇, see also [10, Cor. 3.4]. From $\nabla_v = D_v + A_v$ we obtain when $[v,w] = 0$:

$$R_{vw} = [\nabla_v, \nabla_w] = [D_v, D_w] + D_v A_w - D_w A_v + [A_v, A_w].$$

Here $[D_v, D_w] = R^o$ is the curvature tensor of the sphere $\mathbb{S}^6$,

$$R^o_{vw} x = \langle x, w\rangle v - \langle x, v\rangle w. \tag{29}$$

Now $2A_w = (D_w J)J = S_w J$, hence $2D_v A_w = D_v(S_w J) = (D_v D_w J)J + S_w S_v$. Thus

$$2(D_v A_w - D_w A_v) = [D_v, D_w]J + [S_w, S_v],$$

and moreover

$$4[A_v, A_w] = [S_v J, S_w J] = [S_v, S_w]$$

since $S_v J S_w J = -S_v J J S_w = S_v S_w$. Thus

$$R_{vw} = R^o_{vw} + (1/2)[R^o_{vw}, J] - (1/4)[S_v, S_w]. \tag{30}$$

Since R_o is determined by the metric which is parallel and since J and S are parallel, we see directly that R is parallel.

Lemma 9.2. *For $R_{1\bar 1} := R_{F_1 \bar F_1} = [\nabla_{F_1}, \nabla_{\bar F_1}] = [\nabla', \nabla'']$ we have*

$$\begin{array}{lll} R_{1\bar 1}F_1 = \lambda F_1, & R_{1\bar 1}F_2 = -\frac{\lambda}{2} F_2, & R_{1\bar 1}F_3 = -\frac{\lambda}{2} F_3 \\ R_{1\bar 1}\bar F_1 = -\lambda \bar F_1, & R_{1\bar 1}\bar F_2 = \frac{\lambda}{2} \bar F_2, & R_{1\bar 1}\bar F_3 = \frac{\lambda}{2} \bar F_3. \end{array} \tag{31}$$

Proof. The first line follows from (30) with (29) and (28) where we put $v = F_1$ and $w = \bar F_1$. Applying $R^o_{1\bar 1} = R^o_{F_1 \bar F_1}$ to F_1, F_2, F_3 we observe $\langle F_1, F_j\rangle = 0$ and $\langle F_1, \bar F_j\rangle = \lambda \delta_{1j}$, hence

$$R^o_{1\bar 1}F_1 = \lambda F_1 \quad \text{while} \quad R_{1\bar 1}F_2 = 0, \quad R_{1\bar 1}F_3 = 0\,. \tag{32}$$

In particular, $R_{1\bar{1}}$ commutes with J, and consequently the second term on the right hand side of (30) vanishes, $[R_{1\bar{1}}, J] = 0$. It remains to compute $[S_{F_1}, S_{\bar{F}_1}]$:

$$\begin{aligned} S_{F_1}: &\quad F_2 \mapsto \bar{F}_3, \quad F_3 \mapsto -2\lambda\bar{F}_2, \\ S_{\bar{F}_1}: &\quad \bar{F}_2 \mapsto F_3, \quad \bar{F}_3 \mapsto -2\lambda F_2, \end{aligned}$$

while $F_1, \bar{F}_1$ are mapped to 0. Thus $[S_{F_1}, S_{\bar{F}_1}]$ has eigenvalues -2λ for F_2, F_3 and 2λ for $\bar{F}_2$, $\bar{F}_3$ while $F_1, \bar{F}_1$ are mapped to 0. Now the first line of (31) follows from (30).

For the second line we just observe that $R_{\bar{1}1} = -R_{1\bar{1}}$ and therefore $R_{1\bar{1}}\bar{F}_j = \overline{R_{\bar{1}1}F_j} = -\overline{R_{1\bar{1}}F_j}$. □

10. Structure equations

From Proposition 7.1 and Corollary 9.1 we obtain the derivatives of the Frenet frame:

Proposition 10.1. *Let M be a Riemann surface and $f : M \to \mathbb{S}^6$ a pseudoholomorphic curve. Let ∇ denote the canonical G_2-connection on $\mathbb{S}^6$ and let $\nabla' = \nabla_{\partial/\partial z}$ and $\nabla'' = \nabla_{\partial/\partial \bar{z}}$. Let $F_1 = f_z$, $F_2 = f_{zz}^{\perp}$, $F_3 = f_{\bar{z}} \times f_{\bar{z}\bar{z}}$ be the Frenet frame of f. Then*

$$\begin{array}{rcllll} \nabla' F_1 &=& l_z F_1 & + F_2 & & \\ \nabla' F_2 &=& & m_z F_2 & & + \frac{\mathrm{i}h}{2\lambda\mu} F_3 \\ \nabla' F_3 &=& 0 & & & \\ \nabla'' F_1 &=& 0 & & & \\ \nabla'' F_2 &=& -\frac{\mu}{\lambda} F_1 & & & \\ \nabla'' F_3 &=& & (i\bar{h}/\mu) F_2 & + (l+m)_{\bar{z}} F_3. & \end{array} \tag{33}$$

Corollary 10.1. *The frame $F = (f_z, f_{zz}^{\perp}, f_{\bar{z}} \times f_{\bar{z}\bar{z}})$ of $E_+ = \{v \in \mathbb{O}_c' : f \times v = \mathrm{i}v\}$ solves the differential equations*

$$\nabla' F = F B', \quad \nabla'' F = F B'' \tag{34}$$

with

$$B' = \begin{pmatrix} l_z & 0 & 0 \\ 1 & m_z & 0 \\ 0 & \frac{\mathrm{i}h}{2\lambda\mu} & 0 \end{pmatrix}, \quad B'' = \begin{pmatrix} 0 & -\frac{\mu}{\lambda} & 0 \\ 0 & 0 & \mathrm{i}\bar{h}/\mu \\ 0 & 0 & (l+m)_{\bar{z}} \end{pmatrix}. \tag{35}$$

Remark 1. In the superminimal case $h = 0$ we see that ∇F_3 is a multiple of F_3. In our analogy with the Frenet frame of a space curve c, the third vector F_3 corresponds to the binormal $f_3 = f_1 \times f_2$, where $f_1 = c'$ and $f_2 = (c'')^{\perp}$, and f_3' is proportional to f_3 if and only if the torsion of c vanished (which means that c is a planar curve). Thus Bryant [3] calls superminimal pseudoholomorphic curves *torsion free.*

However, they are not "planar" in any sense: a weak analogue of planes would be a pseudoholomorphic embedding of a complex 2-dimensional manifold into $\mathbb{S}^6$, but there are none. This makes these mappings particularly interesting.

Remark 2. One might wonder why the matrices B', B'' obviously do *not* belong to $\mathfrak{su}_3$ (though this was suggested in section 2). The reason is that the frame F is not normalized. This can easily be corrected by passing to the normalized frame F^o with $F = F^o D$ where $D = \mathrm{diag}(\sqrt{\lambda}, \sqrt{\mu}, \sqrt{2\lambda\mu})$. We have $\nabla F = \nabla(F^o D) = (\nabla F^o)D + F^o \partial D$ and $FB = F^o DB$. Thus from $\nabla F = FB$ we obtain $\nabla F^o = F^o B_o$ with

$$B_o = DBD^{-1} - (\partial D)D^{-1}. \tag{36}$$

We have

$$\begin{aligned}
DB'D^{-1} &= \begin{pmatrix} \sqrt{\lambda} & & \\ & \sqrt{\mu} & \\ & & \sqrt{2\lambda\mu} \end{pmatrix} \begin{pmatrix} l_z & 0 & 0 \\ 1 & m_z & 0 \\ 0 & \frac{ih}{2\lambda\mu} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\lambda}} & & \\ & \frac{1}{\sqrt{\mu}} & \\ & & \frac{1}{\sqrt{2\lambda\mu}} \end{pmatrix} \\
&= \begin{pmatrix} l_z & & \\ \frac{\sqrt{\mu}}{\sqrt{\lambda}} & m_z & \\ & \frac{ih}{\mu\sqrt{2\lambda}} & 0 \end{pmatrix}, \\
(\partial' D)D^{-1} &= \frac{1}{2}\,\mathrm{diag}(l_z, m_z, l_z + m_z)\,, \quad \text{hence by (36),}
\end{aligned}$$

$$B'_o = \begin{pmatrix} \frac{1}{2}l_z & 0 & 0 \\ \frac{\sqrt{\mu}}{\sqrt{\lambda}} & \frac{1}{2}m_z & 0 \\ 0 & \frac{ih}{\mu\sqrt{2\lambda}} & -\frac{1}{2}(l_z + m_z) \end{pmatrix}. \tag{37}$$

Similarly,

$$B''_o = \begin{pmatrix} -\frac{1}{2}l_{\bar z} & -\frac{\sqrt{\mu}}{\sqrt{\lambda}} & 0 \\ 0 & -\frac{1}{2}m_{\bar z} & \frac{i\bar h}{\mu\sqrt{2\lambda}} \\ 0 & 0 & \frac{1}{2}(l_{\bar z} + m_{\bar z}) \end{pmatrix} = -(B'_o)^*\,. \tag{38}$$

Recall that $\nabla_x = \nabla' + \nabla''$ and $\nabla_y = \mathrm{i}(\nabla' - \nabla'')$ where $z = x + \mathrm{i}y$ is the conformal coordinate. Thus

$$\nabla_x F^o = F^o(B'_o + B''_o), \quad \nabla_y F^o = \mathrm{i}F^o(B'_o - B''_o), \tag{39}$$

and the matrices $B'_o + B''_o$ and $\mathrm{i}(B'_o - B''_o)$ belong to $\mathfrak{su}_3$.

11. Integrability conditions

The coefficients of B' and B'' still must satisfy some relations, the *integrability conditions* for the overdetermined system (34). In fact,

$$\nabla'\nabla'' F = \nabla'(FB'') = FB'B'' + FB''_z\,,$$

$$\nabla''\nabla' F = \nabla''(FB') \;\; = \;\; FB''B' + FB'_{\bar z}\,,$$

which implies

$$[\nabla',\nabla'']F = F\left([B',B''] + B''_z - B'_{\bar z}\right).$$

On the other hand we have seen in Lemma 9.2:

$$[\nabla',\nabla'']F \;\; = \;\; R_{1\bar 1}F \;\; = \;\; F\,\mathrm{diag}(\lambda, -\tfrac{\lambda}{2}, -\tfrac{\lambda}{2}).$$

Thus an integrability condition for (34) is

$$\mathrm{diag}(\lambda, -\frac{\lambda}{2}, -\frac{\lambda}{2}) = R_{1\bar 1} = [B',B''] + (B'')_z - (B')_{\bar z}\,. \tag{40}$$

The commutator $[B',B'']$ equals

$$\left[\begin{pmatrix} l_z & 0 & 0 \\ 1 & m_z & 0 \\ 0 & \frac{\mathrm{i}h}{2\lambda\mu} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{\mu}{\lambda} & 0 \\ 0 & 0 & \mathrm{i}\bar h/\mu \\ 0 & 0 & (l+m)_{\bar z} \end{pmatrix}\right]$$

$$= \begin{pmatrix} \frac{\mu}{\lambda} & \frac{\mu}{\lambda}(m_z - l_z) & 0 \\ 0 & -\frac{\mu}{\lambda} + \frac{|h|^2}{2\lambda\mu^2} & \mathrm{i}\bar h\, m_z/\mu \\ 0 & -\frac{\mathrm{i}h}{2\lambda\mu}(l+m)_{\bar z} & -\frac{|h|^2}{2\lambda\mu^2} \end{pmatrix}$$

and the derivatives are

$$(B'')_z = \begin{pmatrix} 0 & -\left(\frac{\mu}{\lambda}\right)_z & 0 \\ 0 & 0 & \mathrm{i}\bar h\left(\frac{1}{\mu}\right)_z \\ 0 & 0 & (l+m)_{\bar z z} \end{pmatrix}, \quad (B')_{\bar z} = \begin{pmatrix} l_{z\bar z} & 0 & 0 \\ 0 & m_{z\bar z} & 0 \\ 0 & \frac{\mathrm{i}h}{2}\left(\frac{1}{\lambda\mu}\right)_{\bar z} & 0 \end{pmatrix}.$$

Since

$$\left(\frac{1}{\mu}\right)_z = -\frac{m_z}{\mu}, \quad \left(\frac{1}{\lambda\mu}\right)_{\bar z} = -\frac{(l+m)_{\bar z}}{\lambda\mu}\,, \quad \left(\frac{\mu}{\lambda}\right)_z = (m-l)_z\,\frac{\mu}{\lambda}\,,$$

we obtain from (40):

$$\begin{aligned} &\mathrm{diag}\left(\lambda, -\frac{\lambda}{2}, -\frac{\lambda}{2}\right) \\ &= \;\; \mathrm{diag}\left(\frac{\mu}{\lambda} - l_{z\bar z},\; \frac{|h|^2}{2\lambda\mu^2} - \frac{\mu}{\lambda} - m_{z\bar z},\; (l+m)_{z\bar z} - \frac{|h|^2}{2\lambda\mu^2}\right). \end{aligned} \tag{41}$$

Lemma 11.1. *Let λ,μ be absolute value type functions on M such that*

$$\mu = \lambda(\lambda + l_{z\bar z}) \tag{42}$$

and let $h : M \to \mathbb{C}$ be a holomorphic function. Then (41) is satisfied if and only if

$$|h|^2 = \lambda^2\mu^2 + 2\lambda\mu^2(l+m)_{z\bar z}\,. \tag{43}$$

Proof. The condition (42) is equivalent to the equality in the first entry, and moreover, the equalities in the second and third entry become the same. The equality in the third entry is (43). □

Lemma 11.2. *If F is the Frenet frame of a pseudoholomorphic curve $f : M \to \mathbb{S}^6$ with Gaussian curvature K and $h = \langle f_{zzz}, f_{zzz} \rangle$, then (43) is equivalent to*

$$|h|^2 = \lambda^6 (1-K)^2 (\Delta \log(1-K) + 1 - 6K) \tag{44}$$

where Δ is the Laplacian of the induced metric on M.

Proof. We have

$$l + m = \log(\lambda\mu) \overset{(18)}{=} \log(\lambda^3(1-K)) = 3\log\lambda + \log(1-K).$$

Further, from $(\log\lambda)_{z\bar z} = -\lambda K$ and $\mu = \lambda^2(1-K)$ (cf. (18)) and $\partial_z \partial_{\bar z} = \frac{1}{2}\lambda\Delta$ we obtain

$$\begin{aligned}
2(l+m)_{z\bar z} &= -6\lambda K + \lambda\Delta\log(1-K) \\
2\lambda\mu^2(l+m)_{z\bar z} &= \lambda^2\mu^2(-6K + \Delta\log(1-K)) \\
\lambda^2\mu^2 + 2\lambda\mu^2(l+m)_{z\bar z} &= \lambda^2\mu^2(1 - 6K + \Delta\log(1-K)) \\
&= \lambda^6(1-K)^2(1 - 6K + \Delta\log(1-K)).
\end{aligned}$$

Thus the conditions (43) and (44) are the same. □

12. Existence of pseudoholomorphic curves

Let $M \subset \mathbb{C}$ be an open domain. Suppose that on M a holomorphic function h and absolute value type functions λ, μ are given satisfying (18) and (43),

$$\begin{aligned}
\mu &= \lambda(\lambda + l_{z\bar z}), \\
|h|^2 &= \lambda^2\mu^2 + 2\lambda\mu^2(l+m)_{z\bar z},
\end{aligned}$$

where $l = \log\lambda$ and $m = \log\mu$. Over M we consider the trivial vector bundle $E = M \times \mathbb{O}'_c$ with a connection ∇ defined by

$$\nabla' F = FB' \text{ and } \nabla'' F = FB''$$

where B', B'' are given in (35),

$$B' = \begin{pmatrix} l_z & 0 & 0 \\ 1 & m_z & 0 \\ 0 & \frac{\mathrm{i}h}{2\lambda\mu} & 0 \end{pmatrix}, \quad B'' = \begin{pmatrix} 0 & -(\lambda + l_{z\bar z}) & 0 \\ 0 & 0 & \mathrm{i}\bar h/\mu \\ 0 & 0 & (l+m)_{\bar z} \end{pmatrix},$$

and where $F = (F_1, F_2, F_3)$ with

$$\begin{aligned}
F_1 &= \sqrt{\lambda}\, F_1^o, & F_2 &= \sqrt{\mu}\, F_2^o, & F_3 &= \sqrt{2\lambda\mu}\, F_3^o, \\
F_1^o &= (i - \mathrm{i}li)/\sqrt{2}, & F_2^o &= (j - \mathrm{i}lj)/\sqrt{2}, & F_3^o &= (k - \mathrm{i}lk)/\sqrt{2}.
\end{aligned}$$

In particular, the only nonzero derivatives are

$$\begin{array}{rcl}\nabla' F_1 &=& l_z F_1 + F_2\,,\\ \nabla' F_2 &=& m_z F_2 + \frac{ih}{2\lambda\mu} F_3\,,\\ \nabla'' F_2 &=& -(\lambda + l_{z\bar z})\, F_1\,,\\ \nabla'' F_3 &=& \frac{i\bar h}{\mu} F_2 + (l+m)_{\bar z} F_3\,.\end{array} \tag{45}$$

On E we have the tensor fields J, S, T, R where

$$Jv = l \times v\,, \quad S_v w = (v \times w)^T$$

with $T = \mathrm{Span}\,(i, j, k, li, lj, lk)$, and where T, R are given by (26), (30), (29). In order to apply the existence and uniqueness theorem in [6] we need that ∇ is a metric connection and J, S (and hence T, R) are parallel with respect to ∇. This follows by passing to the normalized frame F^o and using that $B_o' + B_o''$ and $\mathrm{i}(B_o' - B_o'')$ belong to the Lie algebra $\mathfrak{su}_3$ acting on $\mathbb{C}^3 = T$ with l as complex structure, see (37), (38) (Remark 2 in section 10). The holonomy group SU_3 preserves the metric and the tensors J and S, hence R.

We are ready now to reprove Hashimoto's result [9].

Theorem 12.1. *Let M be a simply connected Riemann surface carrying a compatible Riemannian metric ds^2, possibly with branch points,*[11] *and a holomorphic 6-form Λ. Let K be the Gaussian curvature and Δ the Laplacian of ds^2. Suppose that $1-K$ is an absolute value type function.*

Then there is a unique pseudoholomorphic curve $f : M \to \mathbb{S}^6$ (up to translation with elements of G_2) such that ds^2 is the induced metric and $\Lambda = \Lambda_3$ is the third Hopf diffential (see section 6) if and only if

$$(1-K)^2(\Delta \log(1-K) + 1 - 6K) = |\Lambda|^2\,. \tag{46}$$

Proof. "$\Rightarrow$" If such a pseudoholomorphic curve $f : M \to \mathbb{S}^6$ is given, then (46) is satisfied by Lemma 11.1 and (44), note that

$$|\Lambda|^2 = |h|^2/\lambda^6. \tag{47}$$

"$\Leftarrow$": Let (M, ds^2) and Λ be given with (46). Choosing a conformal coordinate z on some simply connected open subset $M_o \subset M$, we have $ds^2 = 2\lambda dz d\bar z$ for some absolute value type function λ, and $\Lambda = h(z)dz^6$

[11] A compatible Riemannian metric of a Riemann surface is locally of the type $dz^2 = 2\lambda dz d\bar z$ for some conformal coordinate z on M, where λ is a positive function. If we allow for isolated zeros of λ such that λ is an absolute value type function, such zeros are called *branch points* of the metric ds^2.

for some holomorphic function h with (47). Further we put $l = \log\lambda$ and define the absolute value type function

$$\mu = \lambda(\lambda + l_{z\bar{z}}) \overset{(18)}{=} \lambda^2(1-K)\,.$$

Using these functions, we consider the bundle $E = M_o \times T$ with $T = \mathrm{Span}_{\mathbb{C}}(i,j,k,il,jl,kl)$ with sections F_1, F_2, F_3 and a connection ∇ as defined in (45) at the beginning of this section. By the main theorem of [6], there exists a smooth map $f : M \to \mathbb{S}^6$ and a bundle isomorphism $\Phi : E \to f^*TS$ preserving the metric and the tensors J, S, R such that

$$\Phi \circ f_z = F_1 \tag{48}$$

if and only if

$$\begin{array}{rclcl} \nabla'\bar{F}_1 - \nabla''F_1 & = & T(F_1,\bar{F}_1) & = & 0\,, \\ {[\nabla',\nabla'']}F & = & R_{F_1\bar{F}_1}F & = & F\,\mathrm{diag}(\lambda, -\frac{\lambda}{2}, -\frac{\lambda}{2})\,. \end{array} \tag{49}$$

The first equation holds by (45) since $\nabla'\bar{F}_1 = 0 = \nabla''F_1$.

The second equation comes down to (40) and (41) which in turn is equivalent to (44) or (46), by Lemma 11.1. This proves existence and uniqueness of a pair of maps (f,Φ) satisfying (48), and f is pseudo-holomorphic since F_1 and f_z lie in the i-eigenspace of J. Moreover, $F = (F_1, F_2, F_3)$ becomes the Frenet frame along f (via Φ), using (45). In particular, from the "$\Rightarrow$"-part we see $h = \langle (F_2)_z, (F_2)_z\rangle$, cf. (19). This finishes the proof. □

Remark. Replacing Λ by $e^{i\theta}\Lambda$ for some constant angle θ does not change the condition (46). This gives the associated family of the minimal surface f which also consists of pseudoholomorphic curves.

Corollary 12.1. *Let (M, ds^2) be as in the assumptions of the Theorem 12.1. Then there is a superminimal ("torsion free") pseudoholomorphic curve $f : M \to \mathbb{S}^6$, unique up to translations in G_2, with induced metric ds^2 if and only if*

$$\Delta \log(1-K) = 6K - 1. \tag{50}$$

13. Pseudoholomorphic curves in $\mathbb{S}^5$

Another interesting special case is when a pseudoholomorphic curve $f : M \to \mathbb{S}^6$ actually takes values in some equator sphere $\mathbb{S}^5 \subset \mathbb{S}^6$. We will call it a *pseudoholomorphic curve in* $\mathbb{S}^5$.

Lemma 13.1. *Let $f : M \to \mathbb{S}^6$ be a pseudoholomorphic curve and z a conformal coordinate on M. Then f takes values in some great sphere $\mathbb{S}^5 \subset \mathbb{S}^6$ if and only if*

$$|h| = \lambda\mu. \tag{51}$$

Proof. Assume that f lies in $\mathbb{S}^5$. Then there exists a constant unit vector ξ (inside N_2) such that $\langle f, \xi\rangle = 0$. Using $f_z, f_{zz}, f_{zzz} \perp \xi$ and (21) we obtain

$$\lambda\mu\langle f_z \times f_{zz}, \xi\rangle = h\langle f_{\bar z} \times f_{\bar z\bar z}, \xi\rangle$$

and by conjugation

$$\lambda\mu\langle f_{\bar z} \times f_{\bar z\bar z}, \xi\rangle = \bar h\langle f_z \times f_{zz}, \xi\rangle.$$

Multiplying these two equations we find $|h| = \lambda\mu$.

Conversely, we assume that $|h| = \lambda\mu$. Then comparing (21) and its conjugate we obtain a linear relation between $\left(\left(f^\perp_{\bar z\bar z}\right)_{\bar z}\right)^{N_2}$ and its conjugate:

$$\frac{h}{\lambda\mu}\left(\left(f^\perp_{\bar z\bar z}\right)_{\bar z}\right)^{N_2} = \left(\left(f^\perp_{zz}\right)_z\right)^{N_2}. \tag{52}$$

Thus the real and the imaginary part of $((f^\perp_{zz})_z)^{N_2}$ are linearly dependent, and hence there is a real unit vector $\xi \in N_2$ which is perpendicular to $((f^\perp_{zz})_z)^{N_2}$. Consequently, ξ is perpendicular to all derivatives of f up to third order, and hence $\xi_z \perp f, f_z, f_{\bar z}, f_{zz}, f_{\bar z\bar z}, \xi$. So ξ_z must be a multiple of $((f^\perp_{zz})_z)^{N_2}$, and by (52) the same holds for $\xi_{\bar z}$. On the other hand, $\langle \xi_{\bar z}, (f^\perp_{zz})^{N_2}_z\rangle = \langle \xi_{\bar z}, (f^\perp_{zz})_z\rangle = -\langle \xi, (f^\perp_{zz})_{z\bar z}\rangle = 0$ since from $f^\perp_{zz} = f_{zz} + l_z f_z$ we obtain $(f^\perp_{zz})_{z\bar z} = f_{zzz\bar z} + (l_z f_z)_{z\bar z} \in \mathrm{Span}\,(f, f_z, f_{zz}) \perp \xi_z$. Thus ξ is a constant vector and we conclude that f lies in $\mathbb{S}^5 = \mathbb{S}^6 \cap \xi^\perp$. □

Theorem 13.1. *Let M be a simply connected Riemann surface with compatible metric ds^2 (possibly with branch points), and let K be its Gaussian curvature and Δ its Laplacian. Suppose that $1-K$ is an absolute value type function. Then there is an isometric pseudoholomorphic map $f : M \to \mathbb{S}^5$ if and only if*

$$\Delta\log(1-K) = 6K. \tag{53}$$

In fact, up to translations with elements of G_2 there is precisely one associated family of such maps.

Proof. If $f : M \to \mathbb{S}^5$ is pseudoholomorphic with induced metric $ds^2 = 2\lambda dz d\bar z$, we have $|h| = \lambda\mu$ and $|h|^2 = \lambda^2\mu^2 = \lambda^6(1-K)^2$ using $\mu = \lambda^2(1-K)$. Thus the integrability condition

$$\lambda^6(1-K)^2(\Delta\log(1-K) + 1 - 6K) = |h|^2 \tag{44}$$

becomes (53). Conversely, (53) becomes (44) when we put $|h| := \mu\lambda = \lambda^3(1-K)$. Then

$$\Delta\log|h| = 3\Delta\log\lambda + \Delta\log(1-K) = 0,$$

using (53) and the relation between conformal factor and curvature, $\Delta \log \lambda = -2K$. Thus $\log |h|$ is harmonic, hence the real part of a holomorphic function, and $|h|$ is the absolute value of a holomorphic function h, uniquely determined up to some constant phase factor $e^{i\theta}$. Thus $\Lambda = h dz^6$ defines a holomorphic 6-form, and we conclude from Theorem 12.1 that there is a pseudoholomorphic map $f : M \to \mathbb{S}^6$ with induced metric ds^2. Since $|h| = \lambda\mu$, we see from Lemma 13.1 that f takes values in some great sphere $\mathbb{S}^5 \subset \mathbb{S}^6$. $\square$

References

[1] F. Belgun, A. Moroianu: Nearly Kähler 6-manifolds with reduced holonomy, Ann. Glob. Anal. Geom. 19 (2001), 307 - 319

[2] J. Bolton, L. Vrancken, L. Woodward: *On almost complex curves in the nearly Kähler 6-sphere*, Quart. J. Math. 45 (1994), 407 - 427.

[3] R. Bryant: *Submanifolds and special structures on the octonians*, J. Diff. Geom. 17 (1982), 185 - 232

[4] E. Calabi: *Minimal immersions of surfaces in euclidean spheres*, J. Diff. Geom. 1 (1967), 111 - 125

[5] J.-H. Eschenburg, I.V. Guadalupe, R. Tribuzy: *The fundamental equations of minimal surfaces in* $\mathbb{CP}^2$, Math. Ann. 270 (1985), 571 - 598

[6] J.-H. Eschenburg, R. Tribuzy: *Existence and uniqueness of maps into affine homogeneous spaces*, Rend. Sem. Mat. Univ. Padova 69 (1993), 11 - 18

[7] J.-H. Eschenburg, R. Tribuzy: *Constant mean curvature surfaces in 4-space forms*, Rend. Sem. Mat. Univ. Padova 79 (1988), 185 - 202

[8] R. Harvey, H. B. Lawson Jr.: *Calibrated geometries*, Acta Math. 148 (1982), 47 - 157

[9] H. Hashimoto: *J-Holomorphic Curves of a 6-Dimensional Sphere*, Tokyo J. Math. 23 (2000), 137 - 159

[10] F. Lübbe, L. Schäfer: *Pseudo-holomorphic curves in nearly Kähler manifolds*, Diff. Geom. and its Appl. 36 (2014), 24 - 43

[11] G. Ricci: *Lezioni sulla teoria delle superficie*, Verona-Padua 1898, http://ebooks.library.cornell.edu/cgi/t/text/pageviewer-idx?c=math;cc=math;idno=02980001

[12] Th. Vlachos: *Congruence of minimal surfaces and higher fundamental forms*, manuscripta math. 110 (2003), 77 - 91

[13] Th. Vlachos: *Minimal surfaces, Hopf differential and the Ricci condition*, manuscripta math. 126 (2008), 201 - 230

[14] Th. Vlachos: *Exceptional minimal surfaces in spheres*, manuscripta math. 150 (2016), 73-98. DOI: 10.1007/s00229-015-0792-0

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