GEOMETRY OF POSITIVE CURVATURE
THE WORK OF WOLFGANG MEYER

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Fifteen years ago, Detlef Gromoll gave a talk at Münster in honour of Wolfgang Meyer. He summarized what he considered Wolfgang Meyer's most important achievements by the following four short statements, where M always denotes a complete n-dimensional Riemannian manifold with sectional curvature K and $\tilde{M}$ its universal covering.

[1] "Most" compact M have infinitely many closed geodesics.
[2] Every noncompact M with $K > 0$ is contractible.
[3] There is an exotic 7-sphere with $K \geq 0$.
[4] $\frac{1}{4} - \epsilon \leq K \leq 1$, n odd $\Rightarrow \tilde{M} \approx \mathbb{S}^n$ ("homeomorphic").

[1]-[3] are in joint papers with Detlef Gromoll, [4] with Uwe Abresch. I would like to add another result which puts [4] into a context:

[5] $\frac{1}{4} < K < 1 \Rightarrow \tilde{M} \approx \mathbb{S}^n$.

This is the famous sphere theorem of Marcel Berger and Wilhelm Klingenberg which was written up extremely carefully by Detlef Gromoll and Wolfgang Meyer. Their paper became the Springer Lecture Notes volume 55, "Riemannsche Geometrie im Großen", "Riemannian geometry in the Large", which is almost the title of this conference. Every differential geometer of my generation who knew some German learned Riemannian geometry from this textbook.

The main theorem in [1] was also influenced by another research area of Wilhelm Klingenberg: closed Geodesics and the Morse theory of the free loop space of a compact manifold. Closed geodesics are the critical points of the energy function on the space LM of free loops $\lambda : \mathbb{S}^1 \to M$. By Morse theory, the number of critical points of index k is bounded from below by the k-th Betti number b_k. However, every closed geodesic is covered by infinitely many others, the iterates of the given one. These unwanted critical points have to be excluded. Therefore $b_k(LM)$ must be quite large in order to produce enough critical points even without iterates, i.e. enough simply closed geodesics. By the result of [1] it suffices to assume that the sequence $b_k(LM)$ is unbounded. This is true for "most" compact manifolds M (except some

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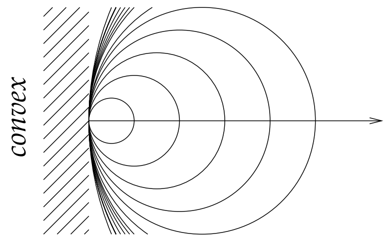
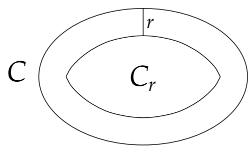
of the simplest ones likes spheres and projective spaces). When M satisfies this topological property, then *every* Riemannian metric on M has infinitely many simply closed geodesics.

The article [2] contained in some sense the noncompact analogue of the sphere theorem. It introduced an idea which later became extremely influential even for Aleksandrov geometry: the link between nonnegative curvature and convexity. For *nonpositive* curvature Riemannian balls are *convex*, at least in the simply connected case. For *nonnegative* curvature, large balls are "virtually" (see below) *concave*, that is their complements are virtually convex, and this holds without any restrictions since a possible cut locus would only reinforce this phenomenon. E.g. in a sphere with $K = 1$, a ball of radius $> \pi/2$ centered at the north pole reaches the southern hemisphere, and thus its complement is convex. If we only assume $K > 0$ or even weaker $K \geq 0$, we might not be able to find a ball of finite radius whose complement is actually convex, but it holds for limits of balls with radius $r \to \infty$ (this is what we mean by "virtually"), the so called *horo-balls* which before have been used only in the geometry of nonpositive curvature.

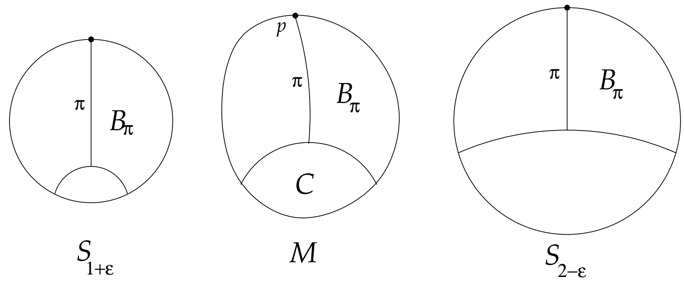


Further, if $C \subset M$ is convex, its interior distance sets $C_r = \{x \in C : |x, \partial C| \geq r\}$ are also convex, even strictly convex when $K > 0$ or when C is strictly convex.



In a complete noncompact manifold there are always geodesic rays, and each of them defines a horoball. The intersection C of the horoball complements for all rays starting in a compact region is a compact convex set whose interior distance sets are strictly convex and thus C shrinks to a point p which is itself comvex: there are no geodesic loops at p. Thus the loop space ΩM is contractible by Morse theory (no critical points for the energy on ΩM other than the constant geodesic). In particular, all homotopy groups are trivial and M is contractible.

As was pointed out later by Micha Gromov, the convexity idea of [1] also works for the sphere theorem [5] (where it is hidden in Toponogov's triangle theorem) and immediately explains the meaning of the curvature bounds $\frac{1}{4}$ and 1. Since the inequalities are strict, the manifold lies "between" two round spheres of radii $1+\epsilon$ (curvature $1-\delta$) and $2-\epsilon$ (curvature $\frac{1}{4}+\delta$). Then the closed ball B_π of radius π in M is immersed as in the small sphere and strictly concave as in the large sphere.



However, B_π is only immersed and the concavity has to be understood in a local sense. It needs a careful analysis of the shrinking of locally strictly convex hypersurfaces in an ambient space with $K \geq 0$ and $n \geq 3$ (in place of Klingenberg's injectivity radius estimate) to see that C is an immersed ball, too, and hence M is covered by the union of two balls, glued together by a diffeomorphism between their boundaries. Thus $\tilde{M}$ is *homeomorphic* to a sphere; "*diffeomorphic*" was out of reach at that time and had to wait for Perelman's ideas.[1]

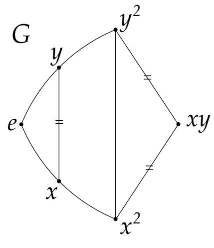
The sphere theorem is beautiful because of its natural curvature bounds which are sharp since the projective spaces over $\mathbb{C}, \mathbb{H}, \mathbb{O}$ satisfy $\frac{1}{4} \leq K \leq 1$. In fact, Marcel Berger added the equality discussion: When $\frac{1}{4} \leq K \leq 1$ and M is simply connected but not a homotopy sphere, then it is even *isometric* to a projective space. However, these spaces live in even dimensions, and Berger noticed that in *odd* dimensions the lower curvature bound can be made even smaller.[2] The proof was indirect using Gromov's compactness theorem, hence no explicit lower bound $\delta < \frac{1}{4}$ could be given. Wolfgang Meyer and Uwe Abresch, using much more refined Jacobi field estimates, found such bounds which are independent of M, even independent of $\dim M$.

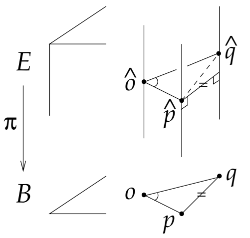
By the proof of the sphere theorem (and the lack of the differentiable version) the question was natural whether a proper homotopy sphere

[1]Brendle, Simon; Schoen, Richard: Manifolds with 1/4-pinched curvature are space forms, Journal of the American Mathematical Society 22 (2009), 287 - 307

[2]Berger, Marcel: Sur les variétés riemanniennes pincées juste au-dessus de $\frac{1}{4}$, Ann. Inst. Fourier 33 (1963), 135 - 150

could satisfy $\frac{1}{4} < K < 1$. But no metric with (at least) $K \geq 0$ was known on any homotopy sphere. In [4], such a metric was constructed using *biquotients* first time. It was known that Lie groups G with biinvariant metrics have $K \geq 0$ and that Riemannian submersions $\pi : E \to B$ are curvature increasing. Here are synthetic proofs.[3]





In [4], the total space E is the group $G = Sp_2$ with its biinvariant metric. Then $G \times G$ acts by isometries on G. For any closed subgroup $U \subset G \times G$ there is an orbital submersion $\pi : G \to G/U$, and G/U is a smooth manifold if the action is *free*. This means for every $(u_1, u_2) \in U \setminus \{(e,e)\}$ that $u_1 g u_2^{-1} \neq g$ for all $g \in G$, in other words, u_2 is not conjugate to u_1 in G. Gromoll and Meyer choosed $U = \{(\left(\begin{smallmatrix} q & \\ & q \end{smallmatrix}\right), \left(\begin{smallmatrix} q & \\ & 1 \end{smallmatrix}\right)) : q \in Sp_1\}$ which acts freely since $\left(\begin{smallmatrix} q & \\ & 1 \end{smallmatrix}\right)$ and its conjugates have a fixed space on $\mathbb{H}^2$ while $\left(\begin{smallmatrix} q & \\ & q \end{smallmatrix}\right)$ does not. U can be enlarged to $V \cong (Sp_1)^2$ where $\left(\begin{smallmatrix} q & \\ & 1 \end{smallmatrix}\right)$ is replaced with $\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)$ for arbitrary $p \in Sp_1$. Since $Sp_2/Sp_1 = \mathbb{S}^7$, we have $Sp_2/V = S^7/Sp_1 = \mathbb{HP}^1 = \mathbb{S}^4$ where Sp_1 acts on $\mathbb{S}^7 \subset \mathbb{H}^2$ by left scalar multiplication. Thus $\Sigma^7 = Sp_2/U$ is an Sp_1-bundle over $Sp_2/V = \mathbb{S}^4$, in fact it is Milnor's exotic sphere M^7_3.[4]

References

[1] (with D. Gromoll) *Periodic geodesics on compact riemannian manifolds*, J. Differential Geometry, 3 (1969), S. 493 - 510

[2] (with D. Gromoll) *On complete open manifolds of positive curvature*, Ann. of Math. 90 (1969), 75 - 90

[3] (with D. Gromoll) *An exotic sphere with nonnegative sectional curvature*, Ann. of Math. 100 (1974), 401 - 406

[4] (with U. Abresch) *A sphere theorem with a pinching constant below 1/4*, J. Differential Geometry 44 (1996), 214 - 261

[5] (with D. Gromoll, W. Klingenberg) Riemannsche Geometrie im Großen, Springer Lecture Notes in Mathematics 55, 1968, 1975

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[3] We have $|e,x^2| = 2|e,x|$ and $|e,y^2| = 2|e,y|$, but $|x^2,y^2| \leq |x^2,xy| + |xy,y^2| = 2|x,y|$ (left figure). Lifting the geodesics *op* and *oq* in B horizontally with starting point $\hat{o} \in \pi^{-1}(o)$ we obtain endpoints $\hat{p}, \hat{q} \in E$ with $|p,q| \leq |\hat{p},\hat{q}|$ since $|\hat{p},\hat{q}|$ may be larger than the distance between the two fibres containing $\hat{p}$ and $\hat{q}$ (right figure).

[4] J.W. Milnor: On manifolds homeomorphic to the 7-sphere, Ann. of Math. 64 (1956), 399 - 405, in particular p. 402.