FROM THE CATENOID-HELICOID DEFORMATION TO THE GEOMETRY OF LOOP GROUPS

J.-H. ESCHENBURG

Abstract. Infinite dimensional methods are used to solve a geometric problem in finite dimensions, the classification of pluriharmonic maps into symmetric spaces. These infinite dimensional objects in turn can be understood from a geometric point of view.

1. Minimal surfaces

Surfaces in euclidean 3-space which locally minimize area (*minimal surfaces*) are beautiful objects, attractive even for modern architecture: their principal curvatures balance each other, i.e. the curvature average (mean curvature) is zero. The reason is simple: deforming the surface into the mean curvature direction would decrease area. Since soap films try to minimize area, minimal surfaces can be realized by a soap film spanned into a (non-planar) closed wire. The oldest known examples are the catenoid (surface of revolution whose profile curve is the cosh-graph) where a soap film is spanned between two coaxial planar circles, and the helicoid (the ruled surface bounded by a double helix), realized by a soap film between two helices.

Catenoid and helicoid belong to a famous one-parameter family (deformation) of surfaces enjoying the following properties:[1]

- the deformation is isometric, i.e. interior distance is preserved,
- the surface remains minimal during deformation,
- every tangent plane is parallel translated,
- the principal curvature directions are rotated,
- the deformation is periodic,
- after a half period a mirror image of the same surface arises ("twist").

We can mimic this deformation using a belt with two sides of different colour. The closed belt approximates the equatiorial region of the catenoid. Moving the left end up (resp. down) and the right end down (resp. up) we obtain an approximation of a helicoid (resp. its mirror

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[1] http://en.wikipedia.org/wiki/Catenoid

image). Such deformations exist for every minimal surface; they are called *associated families*.

2. Harmonic and pluriharmonic maps

Analytically, a surface is an immersion $f : M \to P$ where M is a simply connected Riemann surface (one-dimensional complex manifold) and $P = \mathbb{R}^3$ euclidean 3-space. Choosing a suitable parametrization we may always assume that f is conformal, i.e. the complex structure J is turned into the 90^o-rotation on the tangent plane of f. With such a parametrization f is minimal if and only if it is *harmonic*, $\Delta f = 0$. In fact, associated families are related to harmonic maps of surfaces as follows. Given any smooth map $f : M \to P$ where $P = \mathbb{R}^n$ is euclidean space of any dimension, an *associated family* of f is a smooth variation $f_\theta : M \to P$, $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, such that

$$df_\theta = \Phi_\theta \circ df \circ R_\theta. \tag{1}$$

Here $R_\theta = (\cos\theta)I + (\sin\theta)J$ denotes the rotation by the angle θ in the tangent plane of M and $\Phi_\theta(x)$ is the parallel translation in P from $f(x)$ to $f_\theta(x)$ for any $x \in M$. We can extend this notion without any change to the case when M is a complex manifolds of arbitrary dimension rather than a Riemann surface. Observe from (1) that f_θ need not to exist, but if it does, it is unique up to a parallel translation on P. In the special case $\theta = \pi$ we have $R_\pi = -I$ and equation (1) becomes

$$df_\pi = \Phi_\pi \circ (-df). \tag{2}$$

A solution of this equation can be written down immediately:

$$f_\pi = -f = s_o \circ f \tag{3}$$

where $s_o = -I$ is the point reflection at the origin $0 \in P = \mathbb{R}^n$, and $\Phi_\pi(x)$ is the translation from $f(x)$ to $-f(x)$ which is the composition of the point reflections s_o at 0 and $s_{f(x)}$ at $f(x)$:

$$\Phi_\pi(x) = s_o s_{f(x)}. \tag{4}$$

Theorem 1. *A smooth map $f : M \to P$ is pluriharmonic if and only if there exists an associated family for f.*

Recall that a map f on a complex manifold M is called *pluriharmonic* iff $f|C$ is harmonic for any one-dimensional complex submanifold (complex curve) $C \subset M$. The proof of this theorem is easy since we know that a pluriharmonc map $f : M \to P = \mathbb{R}^n$ is just the real part of a holomorphic map $h : M \to \mathbb{C}^n$, and then the associated family is $f_\theta = \mathrm{Re}\,(e^{i\theta}h)$. Vice versa, if (1) is solvable for every θ, in particular for $\theta = \pi/2$, the one-form $df \circ J$ must be closed; this is precisely the pluriharmonicity of f.

3. Symmetric spaces

The story becomes much more interesting when it is generalized to target spaces P which are no longer euclidean but *symmetric*. A Riemannian manifold P with isometry group $\hat{G}$ is called *symmetric* if for any $p \in P$ there is an isometry $s_p \in \hat{G}$, called *point reflection* or *symmetry* at p, fixing p and reflecting all geodesics through p, i.e. $s_p(p) = p$ and $(ds_p)_p = -I$. Therefore a symmetric space comes with a map

$$C : P \to \hat{G} \quad p \mapsto s_p. \tag{5}$$

This is called *Cartan embedding* although in general it is only a covering map onto its image; note that a point reflection can have more than one isolated fixed points as we see for the sphere $\mathbb{S}^2$. However we will restrict our attention to those spaces P where C is a true embedding. The image $C(P) \subset \hat{G}$ belongs to the set $\mathrm{Inv}(G) = \{g \in \hat{G} : g^{-1} = g\}$ of involutions in G; this is the fixed set of the inversion map $\tau : g \mapsto g^{-1}$ on $\hat{G}$. More precisely $C(P)$ is a connected component of $\mathrm{Inv}(G)$, the set of those involutions which have an isolated fixed point on P. After choosing a bi-invariant metric on $\hat{G}$, τ becomes an isometry and $C(P) \subset G$ is totally geodesic, being a connected component of the fixed set of an isometry. Quite often one replaces the Cartan embedding C by a left translate. E.g. we may choose a base point $o \in P$ and pass to

$$C_o = s_o C, \quad C_o(p) = s_o s_p \tag{6}$$

which always takes values in the transvection group G; it is the identity component of the fixed point set of the involution

$$\tau^o(g) = s_o g^{-1} g_o \tag{7}$$

4. Pluriharmonic maps into symmetric spaces

All what was said about pluriharmonic maps $f : M \to P$ remains true when P is a symmetric space; only the word "translation" has to be replaced with "transvection". Recall that *transvections* on a symmetric space are compositions of an even number of point reflections like translations in euclidean space. But unlike the translation group, the transvection group in general is non-abelian; it is the identity component G of the full isometry group $\hat{G}$ unless P has a euclidean factor with dimension ≥ 2. Moreover, the translation group acts simply transitively which means that there is exactly one translation $\Phi_\theta(x)$ mapping $f(x)$ to $f_\theta(x)$. Not so the transvection group G whose action on P

is transitive but no longer simply transitive: there are many transvections mapping $f(x)$ to $f_\theta(x)$ (they differ by an element of the stabilizer $G_{f(x)}$). This requires an extra condition in order to make a choice, cf. [5, 4]. In fact, $\Phi_\theta(x) \in G$ is determined by its differential at $f(x)$ which is a linear map $T_{f(x)}P \to T_{f_\theta(x)}P$, hence we can view Φ_θ as a section of the bundle $\mathrm{Hom}(f^*TP, f_\theta^*TP)$, and as such a bundle homomorphism it must be *parallel*. In the euclidean case this condition holds trivially for any family of translations $\Phi_\theta(x)$.

From (4) and (6) we obtain

$$\Phi_\pi = C_o \circ f \tag{8}$$

while $\Phi_0 = e$. Hence Φ_θ connects e with the Cartan-embedded version of our pluriharmonic map f (a *solution* of the pluriharmonic map equation). Therefore $\Phi : M \times (\mathbb{R}/2\pi\mathbb{Z}) \to G$ was called *extended solution*.[2] There is a one-to-one correspondence between extended solutions Φ such that Φ_π takes values in $C_o(P)$ and pluriharmonic maps $f : M \to P$ where $C_o \circ f = \Phi_\pi$, cf. [4].

5. Isotropic pluriharmonic maps

A pluriharmonic map $f : M \to P$ is called *isotropic* if its associated family is trivial, $f_\theta = f$ for all θ. For full minimal surfaces or harmonic maps with values in $\mathbb{R}^3$ this is impossible; the reason will become apparent in a moment. However for even dimensions there are examples: holomorphic maps into $\mathbb{R}^{2n} = \mathbb{C}^n$. If f is isotropic, (1) becomes

$$df = \Phi_\theta \circ df \circ R_\theta \tag{9}$$

This shows that $\Phi_\theta(x)$ must fix $f(x)$. Moreover, if f is *full*, i.e. if it does not lie in a proper totally geodesic subspace of P, the group law $R_\theta R_{\theta'} = R_{\theta+\theta'}$ shows that $\Phi(x) : \theta \mapsto \Phi_\theta(x)$ can be chosen to be a one-parameter group as well,

$$\Phi(x) \in \mathrm{Hom}(\mathbb{R}/2\pi\mathbb{Z}\,,\, G_{f(x)}) \tag{10}$$

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[2] This definition using the associated family is due to [4]. The original definition by Ulenbeck [8] is independent of (pluri)harmonic maps. According to [8, 6], an extended solution is a smooth family of maps $\Phi_\theta : M \to G$ such that Φ_π is pluriharmonic and such that the dependence of the $\mathfrak{g}$-valued one-form $\beta_\theta = \Phi_\theta^{-1} d\Phi_\theta$ on θ is very simple: Its $(1,0)$ part (dz-part) is

$$(\beta_\theta)' = \frac{1}{2}(1 - \lambda^{-1})(\beta_\pi)'$$

where $\lambda = e^{-i\theta}$. In fact, Ω carries a complex structure and the latter equation implies that $\Phi : M \to \Omega$ is holomorphic.

where $G_p = \{g \in G : gp = p\}$ is the stabilizer subgroup for any $p \in P$. In particular,

$$df_x = \Phi_\pi(x) \circ (-df_x) = \Phi_\pi(x) \circ s_{f(x)} \circ df_x$$

whence

$$\Phi_\pi(x) = s_{f(x)} = (C \circ f)(x). \tag{11}$$

This differs from (4) only by the left translation with s_o. Obviously, such solutions can exist only if $s_{f(x)} = \Phi_\pi(x)$ is in the identity component $G \subset \hat{G}$ since $\Phi_\pi(x)$ is joined to $\Phi_0(x) = e$ by $\Phi_\theta(x)$. A symmetric space P with the property that a symmetry s_o belongs to the transvection group G is called *inner*; otherwise, if $s_o \in \hat{G} \setminus G$, it is called *outer*.[3] Euclidean space $\mathbb{R}^n$ is inner if and only if $s_o = -I$ has determinant one, i.e. for even n. This explains why there are no full isotropic minimal surfaces in $\mathbb{R}^3$: the twist we have initially observed obstructs it.

As we have seen, the extended solution Φ of an isotropic pluriharmonic map f consists of homomorphisms $\Phi(x) : \mathbb{R}/2\pi\mathbb{Z} \to G$. Due to the parallelity assumption, all these homomorphisms are conjugate to each other, and thus Φ is a (holomorphic) map into a so called *twistor space* Z, the conjugacy class of some homomorphim $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to G$. If f is full, then γ is of a special kind, a so called *canonical* homomorphism, and there are only finitely many such conjugacy classes.[4]

6. Morse theory on the loop space

Isotropic pluriharmonic maps form a subclass which can be rather easily described in terms of finitely many holomorphic functions. We are now ready for our main theorem which is due to [1] for inner symmetric spaces and to [3] for outer ones. It says that if M is compact, the components of the space of pluriharmonic maps are labelled by the isotropic pluriharmonic maps.

[3]A compact symmetric space $P = G/K$ is inner if and only if K contains a maximal torus of G, in other words, if G and K have the same rank. The compact irreducible outer symmetric spaces are precisely SU_n/SO_n, SU_{2n}/Sp_n, odd-dimensional real Grassmannians (including odd dimensional spheres) and E_6/Sp_4, E_6/F_4.

[4]Let $\xi = \gamma'(0)$. Then $\mathrm{Ad}(\gamma(\theta)) = \exp(\theta\,\mathrm{ad}(\xi))$ Since $\mathrm{Ad}(\gamma(2\pi)) = I$, the eigenvalues af $\mathrm{ad}(\xi)$ are in $\mathrm{i}\mathbb{Z}$ where $\mathrm{i} = \sqrt{-1}$. Let $\mathfrak{g}_k$ be the eigenspace of $\mathrm{ad}(\xi)$ corresponding to the eigenvalue $\mathrm{i}k$, $k \in \mathbb{Z}$. It is shown in [5] that $d\Phi$ takes values in the distribution on $Z = \mathrm{Ad}(G)\gamma$ obtained by left translating $\mathfrak{g}_{\pm 1}$; this is the so called superhorizontal distribution. If f is full, the Lie algebra generated by $\mathfrak{g}_{\pm 1}$ must contain every $\mathfrak{g}_k$ for $k \neq 0$, [5]. Such elements $\xi \in \mathfrak{g}$ are called *canonical*; they are precisely the simple sums $\xi_{i_1} + \xi_{i_2} + \dots$ of dual roots $\xi_1, \dots, \xi_r$ of G, see [2]. So there are 2^r canonical elements ξ.

Theorem 2. *Let M be a compact simply connected Kähler manifold, P a compact symmetric space and $f : M \to P$ a pluriharmonic map. Then $f = f^1$ can be deformed by pluriharmonic maps f^t, $t \geq 1$, into an isotropic pluriharmonic map $f^\infty : M \to P$ which is uniquely determined by f.*

In the proof [1] the *loop space* of G is used:

$$\Omega = \Omega G = \{\omega : \mathbb{R}/2\pi\mathbb{Z} \to G;\ \omega(0) = e\}$$

where the loops ω must satisfy a certain regularity (C^∞ or at least H^1). We can consider our extended solution as a map $\Phi : M \to \Omega$. On Ω we have the *energy functional* $E : \Omega \to \mathbb{R}$,

$$E(\omega) = \int_0^{2\pi} |\omega'(\theta)|^2 d\theta.$$

The main idea is: apply the gradient flow of E in order to move $\Phi(M) \subset \Omega$ into a critical manifold of E. As we learned in a first course on differential geometry, the critical points of E are geodesics[5] in G starting and ending at e. Since we have chosen a bi-invariant metric on G, such a geodesic loop is a one-parameter subgroup, hence a group homomorphism $\gamma : \mathbb{R}/2\pi\mathbb{Z} \to G$, and the critical manifolds are the conjugacy classes Ω_γ of such homomorphic circles γ. While usually in differential geometry one tries to *decrease* energy by following the mean curvature flow which is the *negative* gradient flow of E, we are here doing the opposite and follow the *positive* energy gradient which *increases* the energy. Any critical manifold has a finite dimensional domain of attraction U_γ under this flow, the "unstable" manifold under the mean curvature (negative gradient) flow; the dimension of U_γ is just the *index* of the closed geodesic γ. However, most loops are not contained in any such domain. But if M is compact (and simply connected), it has been shown by Uhlenbeck [8] and Ohnita-Valli [6] that any extended solution $\Phi : M \to \Omega$ takes values in the subset of *algebraic loops* $\Omega_{\text{alg}} \subset \Omega$, the loops with finite Fourier expansion,[6] and any such loop does belong to some domain of attraction.[7] In fact, since the closures of these domains are algebraic varieties and Φ is holomorphic, the whole image $\Phi(M)$ lies essentially in one such domain, up to a subvariety of lower dimension which is mapped into the boundary of

[5]Geodesics are locally shortest curves, and the energy is just the square of the length if the curve is parametrized proportional to arc length.

[6]For the Fourier expansion $\omega_\theta = \sum_{k\in\mathbb{Z}} w_k e^{ik\theta} \in G$ we must embed G into a matrix algebra. This is done by the adjoint representation.

[7]From an algebraic view point this fact is the *Bruhat decomposition* of Ω_{alg}, as explained in [7].

that domain. Thus $\Phi(M)$ flows into some well defined Ω_γ, and moreover the flow preserves extended solutions. Thus Φ^∞ is the extended solution of some isotropic pluriharmonic map f^∞. Moreover we may assume that Φ and all Φ^t are invariant under a certain involution of Ω, expressing the fact that f and also all f^t take values in P, see below.

But if P is an *outer* symmetric space, there are no full isotropic pluriharmonic maps into P, so what is f^∞? Recall from (7) that $\Phi^\infty_\pi = C_o \circ f^\infty$ is invariant under $\tau^o(g) = s_o g^{-1} s_o$. This involution τ_o can be extended to an involution T_o of Ω with $(T_o\omega)_\pi = \tau_o\omega_\pi$, namely

$$(T\omega)_\theta = s_o\omega_{\theta+\pi}\omega_\pi^{-1}s_o.$$

Using our freedom of choice for Φ_θ we may assume that Φ is T_o-invariant. Since the energy functional E is also T_o-invariant, the same holds for all Φ^t including Φ^∞. Thus every homomorphism $\gamma = \Phi^\infty(x)$ satisfies

$$\gamma_\theta = s_o\gamma_{\theta+\pi}\gamma_\pi^{-1}s_o = s_o\gamma_\theta s_o,$$

and therefore γ_θ lies in $K = G_o$, the stabilizer group of o, which is the fixed group of the conjugation by s_o. Therefore f^∞ takes values in a subspace $P' \subset P$ which is inner with transvection group K. The corresponding twistor spaces are classified in [3]; their number is 2^s where s is the rank of K (compare footnote 4).

References

[1] F.E. Burstall, M.A. Guest: *Harmonic two-spheres in compact symmetric spaces, revisited.* Math. Ann. 309 (1997), 541 - 572

[2] F.E. Burstall, J.H. Rawnsley: *Twistor Theory for symmetric spaces.* Lect. Notes Math. 1424, Springer-Verlag, Berlin 1990

[3] J.-H. Eschenburg, A.L. Mare, P. Quast: *Pluriharmonic maps into outer symmetric spaces and a subdivision of Weyl chambers*, Bull. London Math. Soc. 42 (2010) 11211133

[4] J.-H. Eschenburg, P. Quast: *The spectral parameter of pluriharmonic maps*, Bull. London Math. Soc. 42 (2010), 229-236

[5] J.-H. Eschenburg, R. Tribuzy: *Associated families of pluriharmonic maps and isotropy*, manuscripta math. 95 (1998), 295 - 310

[6] Y. Ohnita, G. Valli: *Pluriharmonic maps into compact Lie groups and factorization into unitons*, Proc. Lond. Math. Soc. 61 (1990), 546 - 570

[7] A. Presley, G. Segal: *Loop Groups.* Oxford Science Publications, Clarendon Press, 1986

[8] K. Uhlenbeck: *Harmonic maps into Lie groups (classical solutions of the chiral model)*, J.Diff.Geom **30** (1989), 1 - 50

Institut für Mathematik, Universität Augsburg, D-86135 Augsburg, Germany

E-mail address: eschenburg@math.uni-augsburg.de