INDEFINITE EXTRINSIC SYMMETRIC SPACES

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Abstract. We extend Ferus' characterization of extrinsic symmetric spaces to ambient spaces with indefinite inner products.

1. Introduction

Let V be a finite dimensional real vector space with a non-degenerate scalar product (metric) $\langle\ ,\ \rangle$. A linear subspace $W \subset V$ is called *nondegenerate* if $\langle\ ,\ \rangle|_W$ remains non-degenerate (which is trivially satisfied if the inner product is positive definite), otherwise W is called *degenerate.* A smooth submanifold $M \subset V$ is called non-degenerate if so is each of its tangent spaces $T_xM \subset V$, $x \in M$. Then M inherits a semi-Riemannian metric from V. Now for any $x \in M$ we consider the *reflection at the affine normal space* $x + N_xM$ where $N_xM = T_xM^\perp$; this is the affine isometry $s_x : V \to V$ with

$$s_x(x) = x, \quad s_*|_{T_xM} = -I, \quad s_*|_{N_xM} = I.$$

A nondegenerate submanifold $M \subset V$ will be called *extrinsically symmetric* if $s_x(M) = M$ for all $x \in M$. Viewed as a semi-Riemannian manifold with the induced metric, M is a symmetric space since $s_x|_M$ is an isometric point reflection for any $x \in M$. Extrinsic symmetric submanifolds are characterized by the property $\nabla\alpha = 0$ where $\alpha : S^2(TM) \to NM$ is the second fundamental form and $\nabla\alpha : S^3(TM) \to NM$ its covariant derivative. In fact, every extrinsic symmetric submanifold satisfies this property since each s_x preserves $\nabla\alpha_x$ but $s_x = -I$ on $S^3(T_xM)$ (three signs) while $s_x = I$ on N_xM. The converse statement is a theorem of Ferus [4] and Strübing [9].

A rich set of examples is obtained as follows. Let G be a connected Lie group and assume that its Lie algebra $\mathfrak{g}$ is equipped with an $\mathrm{Ad}(G)$-invariant inner product and an orthogonal Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus V$, i.e.

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, V] \subset V, \quad [V, V] \subset \mathfrak{k}. \tag{1}$$

The connected Lie group $K \subset G$ with Lie algebra $\mathfrak{k}$ acts on V by the adjoint action; in fact this is the isotropy representation of the symmetric space G/K. Then an orbit $M = \mathrm{Ad}(K)x$ with $x \in V$ is extrinsic symmetric iff

$$\mathrm{ad}(x)^3 = \lambda\, \mathrm{ad}(x) \tag{2}$$

for some $\lambda \neq 0$ (cf. [5], [3]). If $\lambda = -1$, the extrinsic symmetry at x is $s_x = \exp(\pi\, \mathrm{ad}(x))$. For $\lambda < 0$, this is just a normalization of x; for $\lambda > 0$ we pass to the dual Lie algebra $\mathfrak{g}^* = \mathfrak{k} \oplus iV \subset \mathfrak{g} \otimes \mathbb{C}$ where $i = \sqrt{-1}$. These extrinsic symmetric spaces will be called of *Ferus type* (see below). Most Riemannian symmetric spaces (the so called *symmetric R-spaces*, including Grassmannians, conjugacy classes of real and complex structures, hermitian symmetric spaces and the Lie groups SO_n,

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U_n, Sp_n) can be isometrically embedded as extrinsic symmetric submanifolds of Ferus type.

In general, if $M \subset V$ is extrinsic symmetric, we let

$$\hat{K} = \langle s_x;\ x \in M\rangle \subset O(V) \tag{3}$$

be the group generated by all reflection s_x. The affine map s_x will be called *extrinsic symmetry at* x, and the group $\hat{K}$ is the *symmetry group* of M. We denote its connected component K (the *transvection group*) and its Lie algebra by $\mathfrak{k}$. We call $M \subset V$ *full* if it does not lie in a proper affine subspace, and *indecomposable* if there is no nontrivial orthogonal splitting $V = V_1 \oplus V_2$ and $M = M_1 \times M_2$ with $M_i \subset V_i$.

Dirk Ferus has given the following characterization of extrinsic symmetric spaces in the case where the inner product is positive definite:

Theorem: (Ferus [4],[5]) *Let $M \subset V$ (1) full, (2) indecomposable, (3) extrinsic symmetric. Then we have:*

(A) *The vector space $\mathfrak{g} := \mathfrak{k} \oplus V$ carries the structure of a Lie algebra with Cartan decomposition, and the action of K on V is the the adjoint action of K restricted to $V \subset \mathfrak{g}$.*

(B) *M is the K-orbit of an element $x \in V$ with $\mathrm{ad}(x)^3 = \lambda\,\mathrm{ad}(x)$ for some $\lambda \neq 0$ (i.e. it is of Ferus type).*

It is the aim of the present paper to generalize this theorem to the case where the inner product is *indefinite*. Let us briefly discuss the assumption of Ferus' theorem under this view point. On the one hand, the fullness assumption (1) seems too strong; one would like to discuss the case where M is contained in a proper subspace $W \subset V$ where the inner product is *degenerate* (otherwise we could just pass to W in place of V). However, projecting $M \subset W$ onto the quotient vector space $W/\ker s|_W$ where the induced inner product is nondegenerate, we restore the assumption of the theorem, see Section 2. On the other hand, the indecomposability assumption (2) is much weaker in the indefinite case since it allows for nontrivial (degenerate) K-invariant affine subspaces of V which even may be contained in M. However, we are still able to prove essential parts of Ferus' theorem. The result involves the shape operator (Weingarten map) $S_H(v) = -\partial_v H$ for the mean curvature vector $H = \mathrm{tr}\ \alpha$ where α is the second fundamental form of M, i.e. $\alpha(v,w) = (\partial_v w)^N$.

Theorem A. *Let $M \subset V$ be a (possibly immersed) nondegenerate submanifold which is (1) full, (2) indecomposable, (3) extrinsic symmetric. Then the vector space $\mathfrak{g} := \mathfrak{k} \oplus V$ carries the structure of a Lie algebra with Cartan decomposition, and the linear part of the (affine) action of K on V is the the adjoint action of K restricted to $V \subset \mathfrak{g}$.*

Theorem B. *Under the above assumptions, M is of Ferus type unless $S_H^2 = 0$.*

The present work is based on the 2005 thesis [7] of the first named author. Previously, Naitoh [8] had proved Theorem B by a different method under the additional assumption that there is an umbilic normal vector field. Recently, I. Kath [6] has given a full description of indefinite extrinsic symmetric spaces.

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2. The fullness assumption

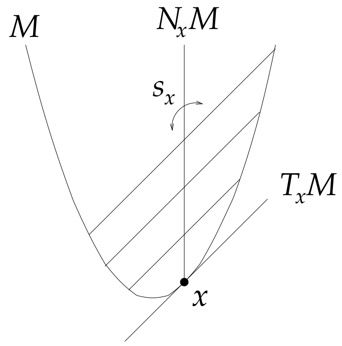
Let us assume that $M \subset V$ is an extrinsic symmetric space lying in some proper linear subspace $W \subset V$. If the inner product is nondegenerate on W, we may replace W by V. The interesting case is when $\langle\ ,\ \rangle|_W$ is degenerate, but still M is nondegenerate. Let

$$N = \ker\langle\ ,\ \rangle|_W = \{n \in W;\ \langle n, w\rangle = 0\ \forall_{w\in W}\}.$$

Then $\bar W = W/N$ inherits a nondegenerate inner product from $\langle\ ,\ \rangle|_W$. Since any tangent space T_xM is nondegenerate, it does not intersect N, and thus the canonical projection $\pi : W \to \bar W$ is an immersion on M. Moreover, π is an isometry, hence it conjugates the reflection at the normal spaces of $M \subset W$ and $\pi(M) \subset \bar W$ which shows that $\pi : M \to \bar W$ is extrinsic symmetric. Hence we have proved:

Theorem 2.1. *Let W be a real vector space with a possibly degenerate inner product $s = \langle\ ,\ \rangle$ and $M \subset W$ a (possibly immersed) extrinsic symmetric submanifold. Let $N = \ker s$ and let $\pi : W \to \bar W = W/N$ be the canonical projection. Then the vector space $\bar W$ inherits a nondegenerate inner product which makes π isometric, and $\pi|_M : M \to \bar W$ is an extrinsic symmetric immersion.*

A simple example is $W = \mathbb{R}^2$ with the inner product $\langle x, y\rangle = x_1y_1$ where we let M be the graph of a parabola, $M = \{(u, tu^2);\ u \in \mathbb{R}\}$ for arbitrary $t \in \mathbb{R}$. For any $x = (u, tu^2) \in M$ we have $T_xM = \mathbb{R}(1, 2tu)$ while N_xM is always the vertical line $N = \mathbb{R}(0, 1)$. The reflection s_x fixes x while its differential $(s_x)_*$ fixes the vector $e_2 = (0, 1)$ and maps $(1, 2tu)$ to $-(1, 2tu)$. Thus for any $y = (v, tv^2) \in M$ we have $s_x(v, tv^2) = (w, tw^2) \in M$ with $w = 2u - v$, hence M is extrisic symmetric. The projection $\pi : (u, v) \mapsto u$ maps M onto the real line (the x_1-axis) with its trivial extrinsic symmetric structure.



Remark. More generally, suppose we have a nondegenerate inner product space $\bar W$ containing an (immersed) extrinsic symmetric submanifold $\bar M \subset \bar W$. Let $W = \bar W \oplus N$ with zero inner product on N, and let $\pi : W \to \bar W$ be the projection onto the first factor. We may ask for all (immersed) extrinsic symmetric $M \subset W$ with $\pi(M) = \bar M$. As the above example suggests, this is a nontrivial problem, cf. [6].

3. Constructing the Lie bracket

From now on we assume fullness. The idea for the proof of Theorem A is to adapt a method of [3] which gives an alternative proof of Ferus' Theorem for positive definite inner products. Let $M \subset V$ be full and extrinsic symmetric and K its symmetry group as above. Fix some $x \in M$. Let $\mathfrak{k} = \mathfrak{k}_+ + \mathfrak{k}_-$ be the Cartan decomposition of $\mathfrak{k}$ corresponding to the extrinsic symmetry s_x, i.e. conjugation with s_x on $\mathfrak{k}$ fixes $\mathfrak{k}_+$ and anti-fixes $\mathfrak{k}_-$. The linear map sending $A \in \mathfrak{k}_-$ onto $Ax \in T_xM$ is an K-equivariant isomorphism between $\mathfrak{k}_-$ and T_xM; its inverse will be called $v \mapsto t_v : T_xM \to \mathfrak{k}_-$. In fact, the map t_v is the *infinitesimal transvection* in v-direction, i.e. the affine isometries $k_v(t) = \exp tt_v \in K$ form the one-parameter group of transvections along the geodesic γ_v whose differential is the parallel transport along γ in both the tangent and normal bundles. Obviously, the linearized action of $\mathfrak{k}_+$ (being the Lie algebra of the isotropy group $K_+ = K_x$) preserves the splitting $V = V_+ + V_-$ where

$$V_+ = N_xM, \quad V_- = T_xM \tag{4}$$

On the other hand, the linearized action of $\mathfrak{k}_-$ reverses this splitting since by differentiating a Levi-Civita-parallel tangent (resp. normal) vector field along γ_v we obtain a normal (resp. tangent) vector field. More precisely, for any $v, w \in V_-$ and $\xi \in V_+$ we have

$$t_v w = \alpha(v, w), \quad t_v \xi = -S_\xi(v) \tag{5}$$

where $S_\xi : V_- \to V_-$ denotes the *shape operator* or *Weingarten map* defined by

$$S_\xi(v) = -(\partial_v \xi)^T, \quad \langle S_\xi(v), w\rangle = \langle \alpha(v, w), \xi\rangle. \tag{6}$$

Since the linear maps $t : \mathfrak{p}_- \to \mathfrak{k}_-$, $v \mapsto t_v$ and $S : V_+ \to S(V_-)$, $\xi \mapsto S_\xi$ are equivariant with respect to the action of $\mathfrak{k}_+$, we have for all $A \in \mathfrak{k}_+$:

$$[A, t_v] = t_{A_* v}, \quad [A_*, S_\xi] = S_{A_* \xi}. \tag{7}$$

First we construct a certain $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}$. Using the K_+-equivariant isomorphism $T : V_- \to \mathfrak{k}_-$ we may transplant the inner product on V_- to $\mathfrak{k}_-$:

$$\langle t_v, t_w\rangle_{\mathfrak{k}_-} = \langle v, w\rangle. \tag{8}$$

This is extended to all of $\mathfrak{k}$ by declaring $\mathfrak{k}_+ \perp \mathfrak{k}_-$ and defining the metric on $\mathfrak{k}_+$ as follows:

$$\langle A, [t_v, t_w]\rangle_{\mathfrak{k}_+} = \langle [A, t_v], t_w\rangle_{\mathfrak{k}_-} = \langle Av, w\rangle, \tag{9}$$

for any $v, w \in T_xM$. In fact, according to the subsequent lemma we have $[\mathfrak{k}_-, \mathfrak{k}_-] = \mathfrak{k}_+$, hence $[t_v, t_w]$ is a generic element of $\mathfrak{k}_+$. We must show that the metric is well defined by (9), in other words that $\langle Av, w\rangle$ only depends on $B := [t_v, t_w]$. We will use the general formula

$$[t_a, t_b]c = -R(a, b)c \tag{10}$$

for the curvature tensor R of the symmetric space M. Putting $A = [t_a, t_b]$ for some $a, b \in T_xM$ we find $\langle Av, w\rangle = \langle [t_a, t_b]v, w\rangle = -\langle R(a, b)v, w\rangle = -\langle R(v, w)a, b\rangle = \langle [t_v, t_w]a, b\rangle = \langle Ba, b\rangle$. Clearly, the metric is $\mathfrak{k}_+$-invariant, and from (9) it is also $\mathfrak{k}_-$-invariant since $\langle Av, w\rangle = \langle [A, t_v], t_w\rangle$.

It remains to show that this inner product is non-degenerate. Since the $\mathfrak{k}_-$-part is nondegenerate (being a copy of the V_--part), it is enough to prove nondegeneracy for the $\mathfrak{k}_+$-part. If this is degenerate, there exists a nonzero $A_o \in \mathfrak{k}_+$ with $\langle A_o v, w\rangle = 0$ for all $v, w \in T_xM$, cf. (9), hence $A_o v = 0$ for all v which contradicts to the effectivity of the isotropy action.

Lemma 3.1. (cf. [2]) $[\mathfrak{k}_-, \mathfrak{k}_-] = \mathfrak{k}_+$.

Proof. This is only due to the fact that K^o is generated by the transvections. In fact, let $\mathfrak{k}_1 = \mathfrak{k}_- + [\mathfrak{k}_-, \mathfrak{k}_-]$. From the Cartan relations $[\mathfrak{k}_-, \mathfrak{k}_+] \subset \mathfrak{k}_-$ and $[\mathfrak{k}_-, \mathfrak{k}_-] \subset \mathfrak{k}_+$ we have $[\mathfrak{k}_-, [\mathfrak{k}_-, \mathfrak{k}_-]] \subset [\mathfrak{k}_-, \mathfrak{k}_+] \subset \mathfrak{k}_-$ (Lie triple property) and therefore $\mathfrak{k}_1 \subset \mathfrak{k}$ is a Lie subalgebra. Let $K_1 \subset K$ be the corresponding connected Lie subgroup. We need to show that all transvections $s_y s_z$ for any two $y, z \in M$ belong to K_1. It suffices to show this when y and z are connected by a geodesic γ: In general there is only a geodesic polygon connecting y and z, but labeling its vertices $y = y_0$, $y_1, \dots, y_k = z$, we have $s_y s_z = s_{y_0} s_{y_1} s_{y_1} s_{y_2} \dots s_{y_{k-1}} s_{y_k}$ and hence it suffices to show $s_{y_{i-1}} s_{y_i} \in K_1$.

Thus we assume that y and z lie on a common geodesic γ with (say) $\gamma(0) = z$ and $\gamma(1/2) = y$. We want to show $s_y s_z \in K_1$. If $z = x$, this is obvious since then $s_y s_x = \exp t_v \subset \exp \mathfrak{k}_-$ where $v = \gamma'(0)$. For an arbitrary geodesic segment $\gamma : [0, 1] \to M$ we put

$$\tau_\gamma = s_{\gamma(1/2)} s_{\gamma(0)}.$$

This is the transvection along γ sending $\gamma(0)$ to $\gamma(1)$. Likewise for a geodesic polygon $p = \gamma_1 * \cdots * \gamma_k$ (concatenation of k geodesic segments) we put $\tau_p = \tau_{\gamma_k} \dots \tau_{\gamma_1}$. By induction over k we claim $\tau_p \in K_1$ for any geodesic polygon p starting at our base point x. This is clear for $k = 1$. In the general case we let $p = p' * \gamma$ where $\gamma = \gamma_k$ and $p' = \gamma_1 * \cdots * \gamma_{k-1}$. By induction hypothesis $\tau_{p'} \in K_1$. Further, the geodesic $\beta = \tau_{p'}^{-1} \circ \gamma$ starts at x, and hence $\tau_\beta \in K_1$ whence $\tau_\gamma = \tau_{p'}^{-1} \tau_\beta \tau_{p'} \in K_1$.

Now let $y, z \in M$ be as above. We have to make sure that $s_y s_z = \tau_\gamma$ is contained in K_1. Join x to z by a geodesic polygon p. Then $\tau_p \in K_1$, and the geodesic $\beta = \tau_p^{-1} \circ \gamma$ starts at x. Thus $\tau_\beta \in K_1$ whence $\tau_\gamma = \tau_p \tau_\beta \tau_p^{-1} \in K_1$. $\square$

Next we define a skew symmetric product $[\ ,\ ]$ on

$$\mathfrak{g} = \mathfrak{k} + V \tag{11}$$

extending the Lie product on $\mathfrak{k}$ as follows. For any $A \in \mathfrak{k}$ and $v, w \in V$ we define $[A, v] \in V$ and $[v, w] \in \mathfrak{k}$ by

$$[A, v] = A_* v, \quad \langle [v, w], A \rangle_{\mathfrak{k}} = \langle Av, w \rangle. \tag{12}$$

Note that

$$[v, w] = [t_v, t_w] = R(v, w), \tag{13}$$

since $\langle A, [t_v, t_w] \rangle = \langle [A, t_v], t_w \rangle = \langle t_{Av}, t_w \rangle = \langle Av, w \rangle$.

By [3], p. 518–519, we have defined a Lie algebra stucture on $\mathfrak{g}$ (the main work amounts to proving the Jacobi identity for all $u, v, w \in V \subset \mathfrak{g}$) with Cartan decomposition (11), and the direct sum metric on $\mathfrak{g} = \mathfrak{k} + V$ is $\mathrm{ad}(\mathfrak{g})$-invariant. Further,

$$\mathfrak{g}_+ := \mathfrak{k}_+ \oplus V_+ \tag{14}$$

is a subalgebra since $\langle [V_+, V_+], \mathfrak{k}_- \rangle_{\mathfrak{g}} = \langle [\mathfrak{k}_-, V_+], V_+ \rangle_{\mathfrak{g}} \subset \langle V_-, V_+ \rangle = 0$. Putting

$$\mathfrak{g}_- = \mathfrak{k}_- \oplus V_-, \tag{15}$$

we have a second Cartan decomposition,

$$\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-. \tag{16}$$

which is compatible to the first one. This finishes the proof of Theorem A.

4. Linearity of the action

Theorem 4.1. *Let $M \subset V$ be full, indecomposable and extrinsic symmetric. Suppose that all affine normal spaces $x + N_x M$, $x \in M$, have a nonempty intersection. Then M is of Ferus type.*

Proof. Up to translations we may assume that the common intersection of the affine normal spaces contains the origin 0. Thus $x \in N_x M$ for all $x \in M$, and the extrinsic symmetries s_x are linear (fixing 0). Hence $\hat{K} = \langle s_x;\ x \in M \rangle$ is a subgroup of the orthogonal group $O(V)$. Now we fix $x \in M$ and let $V_- = T_x M$ and $V_+ = N_x M$. Then for all $v, w \in V_-$ we have

$$\begin{aligned} \mathrm{ad}(x) t_v &= -t_v x = -v, \\ \langle t_w, \mathrm{ad}(x) v \rangle &= \langle t_w x, v \rangle = \langle w, v \rangle = \langle t_w, t_v \rangle. \end{aligned}$$

Further, $\mathfrak{k}_+$ is the stabilizer subalgebra for x, hence for all $A \in \mathfrak{k}_+$ and $\xi \in V_+$ we have $[A, x] = 0$ and $\langle A, [x, \xi] \rangle = \langle [A, x], \xi \rangle = 0$. Thus

$$\mathrm{ad}(x) v = t_v, \quad \mathrm{ad}(x) t_v = -v, \quad \mathrm{ad}(x)|_{\mathfrak{g}_+} = 0 \tag{17}$$

showing $M = \mathrm{Ad}(K) x$ with $\mathrm{ad}(x)^3 = -\mathrm{ad}(x)$ which shows that M has Ferus type. $\square$

Remark. The situation looks very similar to the case of positive definite inner product. However note that x could be a light vector, i.e. $\langle x, x\rangle = 0$. Otherwise, if $\langle x, x\rangle = s \neq 0$, then M is contained in the "sphere" $S_s = \{x \in V;\ \langle x, x\rangle = s\}$, but different from the positive definite case, it need not to be minimal inside S_s.

5. The Killing form

We have constructed a metric Lie algebra $\mathfrak{g} = \mathfrak{k}+V$ equipped with two commuting involutions σ, τ leading to the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{k}_+ + \mathfrak{k}_- + V_+ + V_-. \tag{18}$$

A very important tool is the Killing form B on the Lie algebra $\mathfrak{g}$ which turns out to be closely related to the second fundamental form α of $M \subset V$. Recall that for any $X, Y \in \mathfrak{g}$ we have

$$B(X,Y) = \operatorname{tr}_{\mathfrak{g}} \operatorname{ad}(X)\operatorname{ad}(Y) = \sum_i \epsilon_i \langle \operatorname{ad}(X)\operatorname{ad}(Y)E_i, E_i\rangle_{\mathfrak{g}} \tag{19}$$

where (E_i) is an orthonormal basis of $\mathfrak{g}$, i.e. $\langle E_i, E_j\rangle = \epsilon_i\delta_{ij}$ with $\epsilon_i = \pm 1$. We will choose this basis adapted to the orthogonal splitting (18).

Lemma 5.1. *For any $v, w \in V_-$ and $\xi, \eta \in V_+$ we have*

$$\begin{aligned} B(v,w) &= 2\operatorname{tr}_{V_-}(\operatorname{ad}(v)\operatorname{ad}(w)) + 2\operatorname{tr}_{\mathfrak{k}_-}(\operatorname{ad}(v)\operatorname{ad}(w)),\\ B(t_v,t_w) &= 2\operatorname{tr}_{\mathfrak{k}_-}(\operatorname{ad}(t_v)\operatorname{ad}(t_w)) + 2\operatorname{tr}_{V_-}(\operatorname{ad}(t_v)\operatorname{ad}(t_w)),\\ B(\xi,\eta) &= 2\operatorname{tr}_{\mathfrak{k}_-}(\operatorname{ad}(\xi)\operatorname{ad}(\eta)) + 2\operatorname{tr}_{\mathfrak{k}_+}(\operatorname{ad}(\xi)\operatorname{ad}(\eta)). \end{aligned}$$

Proof. Due to the Cartan relations

$$[V,V] \subset \mathfrak{k}, \quad [\mathfrak{g}_-\mathfrak{g}_-] \subset \mathfrak{g}_+, \tag{20}$$

the linear map $\operatorname{ad}(v)$ is a skew symmetric transformation which maps $V_\pm$ to $\mathfrak{k}_\mp$ and vice versa. Likewise $\operatorname{ad}(t_v)$ interchanges the two subspaces in each of the pairs (V_+, V_-) and $(\mathfrak{k}_+, \mathfrak{k}_-)$, and the same holds for $\operatorname{ad}(\xi)$ and the pairs $(\mathfrak{k}_-, V_-)$ and $(\mathfrak{k}_+, V_+)$. This shows that the partial traces for B are the same on the two components of each pair: E.g. on $V_+ + \mathfrak{k}_-$ we have $\operatorname{ad}(v) = \left(\begin{smallmatrix} & A' \\ A & \end{smallmatrix}\right)$ where $A = \operatorname{ad}(v)|_{V_+}$ and $A' = \operatorname{ad}(v)|_{\mathfrak{k}_-}$, but since $\operatorname{ad}(v)^* = -\operatorname{ad}(v)$, we have $A' = -A^*$. Thus

$$\begin{aligned} \operatorname{tr}_{V_+ + \mathfrak{k}_-}(\operatorname{ad}(v)\operatorname{ad}(w)) &= \operatorname{tr}_{V_+ + \mathfrak{k}_-} \left(\begin{smallmatrix} & -A^* \\ A & \end{smallmatrix}\right)\left(\begin{smallmatrix} & -B^* \\ B & \end{smallmatrix}\right)\\ &= \operatorname{tr}_{V_+ + \mathfrak{k}_-} \left(\begin{smallmatrix} -A^*B & \\ & -AB^* \end{smallmatrix}\right)\\ &= -\operatorname{tr}(A^*B) - \operatorname{tr}(AB^*), \end{aligned}$$

and the latter two terms are equal. Using a similar argument on $V_- + \mathfrak{k}_+$ we obtain

$$\begin{aligned} \operatorname{tr}_{V_+ + \mathfrak{k}_-}(\operatorname{ad}(v)\operatorname{ad}(w)) &= 2\operatorname{tr}_{\mathfrak{k}_-}(\operatorname{ad}(v)\operatorname{ad}(w)),\\ \operatorname{tr}_{V_- + \mathfrak{k}_+}(\operatorname{ad}(v)\operatorname{ad}(w)) &= 2\operatorname{tr}_{V_-}(\operatorname{ad}(v)\operatorname{ad}(w)) \end{aligned}$$

which shows the equality for $B(v,w) = \operatorname{tr}_{\mathfrak{g}}(\operatorname{ad}(v)\operatorname{ad}(w))$. The other two equations follow quite similarly. $\square$

Lemma 5.2. *For any $v, w \in V_-$ we have*

$$B(v,w) = -2\langle \alpha(v,w), H\rangle = B(t_v, t_w) \tag{21}$$

where $H = \operatorname{tr}_{V_-} \alpha$.

Proof. Choosing an orthonormal basis $(e_1, \dots, e_m)$ of V_-, we get

$$\begin{aligned} B(v,w) &= 2\sum \epsilon_i\langle \operatorname{ad}(v)\operatorname{ad}(w)e_i, e_i\rangle + 2\sum \epsilon_i \langle \operatorname{ad}(v)\operatorname{ad}(w)t_{e_i}, t_{e_i}\rangle\\ &= -2\sum \epsilon_i\langle [w,e_i],[v,e_i]\rangle - 2\sum \epsilon_i\langle t_{e_i}w, t_{e_i}v\rangle\\ &= 2\sum \epsilon_i \left\{\langle R(w,e_i)v, e_i\rangle - \langle \alpha(e_i,w), \alpha(e_i,v)\rangle\right\} \end{aligned}$$

$$
\begin{aligned}
& \overset{(G)}{=} && -2\sum \epsilon_i\langle\alpha(e_i,e_i),\alpha(v,w)\rangle = -2\langle H,\alpha(v,w)\rangle,\\
B(t_v,t_w) & = && 2\sum \epsilon_i\langle \mathrm{ad}(t_v)\,\mathrm{ad}(t_w)t_{e_i},t_{e_i}\rangle + 2\sum \epsilon_i\langle \mathrm{ad}(t_v)\,\mathrm{ad}(t_w)e_i,e_i\rangle\\
& = && -2\sum \epsilon_i\langle [t_w,t_{e_i}],[t_v,t_{e_i}]\rangle - 2\sum \epsilon_i\langle t_we_i,t_ve_i\rangle\\
& = && 2\sum \epsilon_i\left\{\langle R(w,e_i)v,e_i\rangle - \langle\alpha(e_i,w),\alpha(e_i,v)\rangle\right\}\\
& \overset{(G)}{=} && -2\sum \epsilon_i\langle\alpha(e_i,e_i),\alpha(v,w)\rangle = -2\langle H,\alpha(v,w)\rangle,
\end{aligned}
$$

where (G) refers to the Gauss equations for $M \subset V$,

$$\langle R(a,b)c,d\rangle = \langle\alpha(a,d),\alpha(b,c)\rangle - \langle\alpha(b,d),\alpha(a,c)\rangle. \tag{G}$$

This finishes the proof. □

Lemma 5.3. *Let $\xi,\eta \in V_+$ with $\xi = \alpha(v,w) = t_vw$. Then*

$$B(\xi,\eta) = -2\langle\alpha(S_\eta v,w),H\rangle. \tag{22}$$

Proof. $B(\xi,\eta) = B(t_vw,\eta) = -B(w,t_v\eta) = B(w,S_\eta v) \overset{(21)}{=} -2\langle\alpha(w,S_\eta v),H\rangle.$ □

Lemma 5.4. *Let $\xi,\eta \in V_+$ with $[\mathfrak{k}_+,\eta] = 0$. Then*

$$B(\xi,\eta) = -2\,\mathrm{tr}_{V_-}(S_\xi S_\eta). \tag{23}$$

Proof. From Lemma 5.1 we obtain

$$B(\xi,\eta) = -2\sum \epsilon_i\langle[\xi,t_{e_i}],[\eta,t_{e_i}]\rangle - 2\sum \epsilon_j\langle[\xi,A_j],[\eta,A_j]\rangle$$

where (A_j) denotes an orthonormal basis of $\mathfrak{k}_+$. But the second term vanishes by the assumption $[\eta,\mathfrak{k}_+] = 0$, hence the claim follows using (5). □

6. The shape operators

Let $M \subset V$ be extrinsic symmetric. As before, we fix some $x \in M$ and let $V_- = T_xM$ and $V_+ = N_xM$. For all normal vectors $\xi \in V_+$ we consider the shape operators S_ξ which are self adjoint endomorphisms of V_-. Their commutators are obtained from the *Ricci equation*

$$\langle R^N(v,w)\eta,\xi\rangle = \langle[S_\xi,S_\eta]v,w\rangle \tag{R}$$

for any $v,w \in V_-$ and $\xi,\eta \in V_+$ where R^N denotes the curvature tensor of the normal bundle. If η can be extended to a normal vector field which is parallel, the left hand side of (R) vanishes, hence the corresponding shape operator S_η commutes with any S_ξ.

Most important among the normal vectors is the *mean curvature vector*

$$H = \mathrm{tr}\ \alpha = \sum \epsilon_i\alpha(e_i,e_i) \tag{24}$$

where $e_1,\dots,e_m$ is an orthonormal basis of V_-, i.e. $\langle e_i,e_j\rangle = \epsilon_i\delta_{ij}$ with $\epsilon_i = \pm 1$. Since α is parallel, so are H and S_H, and hence S_H commutes with any S_ξ (Ricci equation).

Yet there are other parallel normal fields closely related to H, e.g.

$$H_1 := \mathrm{tr}\ \alpha(S_H.,.) = \sum \epsilon_i\alpha(S_He_i,e_i). \tag{25}$$

Lemma 6.1.

$$S_{H_1} = (S_H)^2. \tag{26}$$

Proof. We apply the lemmas 5.3 and 5.4 for $\xi = \alpha(v,w)$ and $\eta = H$. The two expressions for $-B(\xi, H)$ are

$$\begin{aligned}
-B(\xi,H) &= 2\langle \alpha(S_H v, w), H\rangle = \underline{2\langle S_H S_H v, w\rangle},\\
-B(\xi,H) &= 2\sum \epsilon_i \langle S_\xi e_i, S_H e_i\rangle\\
&= 2\sum \epsilon_i \langle \alpha(e_i, S_H e_i), \xi\rangle\\
&= 2\langle H_1, \xi\rangle\\
&= 2\langle H_1, \alpha(v,w)\rangle = \underline{2\langle S_{H_1} v, w\rangle}.
\end{aligned}$$

□

In the case where the metric on V is positive definite, S_H is diagonalizable with constant real eigenvalues λ, and TM splits into the mutually orthogonal eigendistributions E_λ. Since S_H is parallel, so are the E_λ, and any shape operator S_ξ preserves E_λ. Hence by indecomposability (Moore Lemma) there can be only one eigenvalue λ. This must be nonzero since M cannot be minimal (cf. [1]).[1] Now the map $M \ni x \mapsto x + \lambda^{-1}H(x) \in V$ is constant which shows that M is contained in a sphere $S \subset V$ and M is of Ferus type.

In the indefinite case we arrive at the same conclusion when S_H is diagonalizable with real eigenvalues. But this cannot be concluded from the symmetry of S_H anymore. Therefore, as a first step, we replace the eigenspace decomposition by a coarser one: For each eigenvalue $\lambda \in \mathbb{C}$ of S_H we put

$$E_\lambda = \ker(S_H - \lambda I)^k$$

for a sufficiently large integer k. These *generalized eigenspaces* form the Jordan decomposition of $V^c = V \otimes \mathbb{C}$ which by self adjointness of $T = (S_H - \lambda I)^k$ is orthogonal with respect to the complexified inner product.[2] A real decomposition is obtained by combining conjugate pairs of eigenvalues λ and $\bar\lambda$. The subspaces $E_\lambda + E_{\bar\lambda}$ form a real parallel decomposition of TM which again is S_ξ-invariant for any normal vector ξ. Using the lemma of Moore (cf. [1]), we may conclude from the indecomposability that there is just one conjugate pair $\lambda, \bar\lambda$ of eigenvalues of S_H (which of course might be equal).

But we can do better. We consider the space P of all parallel real normal fields on M. Since all shape operators S_η with $\eta \in P$ commute with each other, they preserve each other's generalized eigenspaces, and we can find a simultaneous Jordan decomposition: There is a finite set Λ of linear forms $\lambda : P \to \mathbb{C}$, invariant under complex conjugation ($\lambda \in \Lambda \Rightarrow \bar\lambda \in \Lambda$), and a decomposition

$$V^c_- = \sum_{\lambda\in\Lambda} V_\lambda \tag{27}$$

such that $V_\lambda \subset \ker(S_\eta - \lambda(\eta)I)^k$ for all $\eta \in P$, where $k \in \mathbb{N}$ is sufficiently large. In fact, if $\eta_1, \dots, \eta_r$ is a basis of P, then $V_\lambda = \bigcap_{j=1}^r E_j$ where E_j is the generalized eigenspace of S_{η_j} corresponding to the eigenvalue $\lambda(\eta_j)$. Since S_{η_j} is parallel, the same holds for the generalized eigenspaces E_i and their intersection, thus the decomposition (27) is parallel along M and therefore $\mathfrak{k}_+$-invariant.

[1] One way to see this is using the Grassmann valued Gauss map $\tau : M \to Gr_m(V)$ which assigns to each $x \in M$ its tangent plane $T_xM \subset V$. This is a K-equivariant map with $d\tau = \alpha$. Hence $\ker d\tau$ is an integrable distribution whose leaves form a euclidean factor, but this cannot hold in the indecomposable case. Thus τ is an immersion, and from the extrinsic symmetry we see that $\tau(M)$ is a totally geodesic symmetric submanifold of $Gr_m(V)$. Thus it has nonnegative sectional curvature which is not possible for a minimal submanifold in euclidean space.

[2] The integer k is large enough to make $\ker T$ and im T transversal, and by self adjointness, $\ker T \perp$ im T, hence these subspace form an S_H-invariant orthogonal decomposition.

Moreover, each V_λ is nondegenerate which we see by induction over r: If $r = 1$, then V_λ is a generalized eigenspace of S_η and hence nondegenerate. For $r > 1$ we put $W = \bigcap_{j=1}^{r-1} E_j$. This is nondegenerate by induction hypothesis and invariant under S_{η_r}. Then V_λ is a generalized eigenspace of $S_{\eta_r}|_W$ (considered as a symmetric endomorphism on W), and thus V_λ is a nondegenerate subspace of W.

The decomposition (27) is invariant under all S_ξ, $\xi \in V_+$, and if we combine V_λ and $V_{\bar\lambda}$, we get a real decomposition. Hence we can use again Moore's Lemma and the indecomposability of M to conclude that there is just one such pair, $\Lambda = \{\lambda, \bar\lambda\}$, or in other words

$$V_-^c = V_\lambda + V_{\bar\lambda}. \tag{28}$$

Lemma 6.2. *For any $\eta \in P$ such that $\lambda(\eta) = t \in \mathbb{R}$, we have $S_\eta = tI + N$ where N is nilpotent with $S_H N = 0$.*

Proof. Since $\lambda(\eta) = \overline{\lambda(\eta)} = t$, we have $S_\eta = tI + N$ on $V_-^c = V_\lambda + V_{\bar\lambda}$ with N nilpotent. Take any normal vector $\xi = \alpha(v, w)$. Note that S_ξ commutes with S_η and hence with N, thus $S_\xi N$ is nilpotent ($(S_\xi N)^k = S_\xi^k N^k = 0$) and so it has trace zero. Now from (23) we obtain

$$\begin{aligned} B(\xi,\eta) &= -2\ \mathrm{tr}(S_\xi S_\eta) \\ &= -2t\ \mathrm{tr}\, S_\xi \\ &= -2t \sum \epsilon_i \langle S_\xi e_i, e_i\rangle \\ &= -2t \sum \epsilon_i \langle \xi, \alpha(e_i, e_i)\rangle \\ &= -2t\langle \xi, H\rangle \end{aligned}$$

On the other hand, from (22) we see

$$\begin{aligned} B(\xi,\eta) &= -2\langle \alpha(S_\eta v, w), H\rangle \\ &= -2t\langle \alpha(v,w), H\rangle - 2\langle \alpha(Nv, w), H\rangle \\ &= -2t\langle \xi, H\rangle - 2\langle S_H N v, w\rangle. \end{aligned}$$

Comparing the two results we see that $S_H N = 0$ ☐

Lemma 6.3. *Either $S_H^2 = 0$ or there is some $\eta \in P$ with $S_\eta = tI$ for some nonzero $t \in \mathbb{R}$.*

Proof. Suppose that S_H has a non-real eigenvalue $\lambda(H)$. Then S_H^2 and S_H are linearly independent, and so H_1 and H are linearly independent. Then we find a real nonzero linear combination $\eta = aH + bH_1$ such that $\lambda(\eta) = t$ is real. From the previous lemma we see that $S_\eta = tI + N$ with $S_H N = 0$. But S_H is invertible, hence $N = 0$. We have $S_\eta = tI$ with $t \neq 0$: If $S_\eta = 0$ for any parallel normal field η, we would have $d\eta = 0$ and thus η would be a constant vector with $M \subset \eta^\perp$, but this was excluded by the fullness assumption.

On the other hand, if $\lambda(H) \in \mathbb{R}$, we have $S_H = sI + N$ with $S_H N = 0$, but $S_H N = sN + N^2$. If $N^2 \neq 0$, then N^2 and N are linearly independent and $N^2 + sN = 0$ is impossible. Thus $N^2 = 0$ and $s = 0$. ☐

Now we have finished the proof of Theorem B: M is of Ferus type unless $S_H^2 = 0$. In fact, by Lemma 6.3 the affine normal spaces $x + N_x M$ have a common intersection point $x - \eta(x)/t$, and the result follows from Theorem 4.1.

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