# PENROSE TYPE TILINGS

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Verfasser:

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# Contents

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I would also like to thank my father for helping me to convert an unreadable project into a partially readable thesis.

# Introduction

Beauty is a concept that everyone can relate to, whether it is in a face, a scenic view or maybe a work of art. Everyone has encountered beauty in some shape or form. It is without doubt subjective and strongly dependent on the observer. However one of the few true objective things that is strongly linked to beauty in any form is symmetry. Studies have shown that 'conventionally' beautiful faces are generally symmetrical. In art symmetry or obvious lack of symmetry can be powerful tools.

In ancient times patterns with lots of symmetry and in particular many centers of symmetry were discovered and studied. More recently interesting quasi-symmetrical patterns have been studied. One such pattern is called a 'Penrose tiling'. There are many ways to represent this, one of the most common is to use 'thin' and 'fat' 'diamonds'. Figure 1 is an example of a Penrose tiling with these sort of tiles. A Penrose tiling has many local centers of symmetry (rotation by $\frac{2\pi}{5}$) but at most one global center.

One of the most attractive things about the Penrose tiling is the way it can be presented to a popular audience without going into difficult mathematical detail. My approach is differs from the popular approach. I am trying to work as generally as possible which leads to levels of abstraction that are not presentable in a popular text. Therefore the audience I aim for is one that has a sufficient mathematical background to be comfortable working with abstract concepts.

One construction of a Penrose tiling is to start with an integer lattice in 5 dimensional space and project it onto a 2 dimensional target subspace. The first and main part of my project was to prove in as general a context as possible that this construction results in a tiling of the target space. My approach is formal. If you are unfamiliar with this construction very simple explanations can be found on the web either by searching for 'Penrose tilings' or following some of the links; [10, 8] both give a brief description of the construction.

In the second part of my project I will informally discuss the inflation deflation property that makes the Penrose tiling special. I then give ideas on the generalisation of this property.

The first part may seem over-formal, and hence difficult to follow. The formality was necessary to prove the general result. I have attempted to present the ideas in the second part in a more approachable way and an understanding of the construction from the first part is all that should be needed.

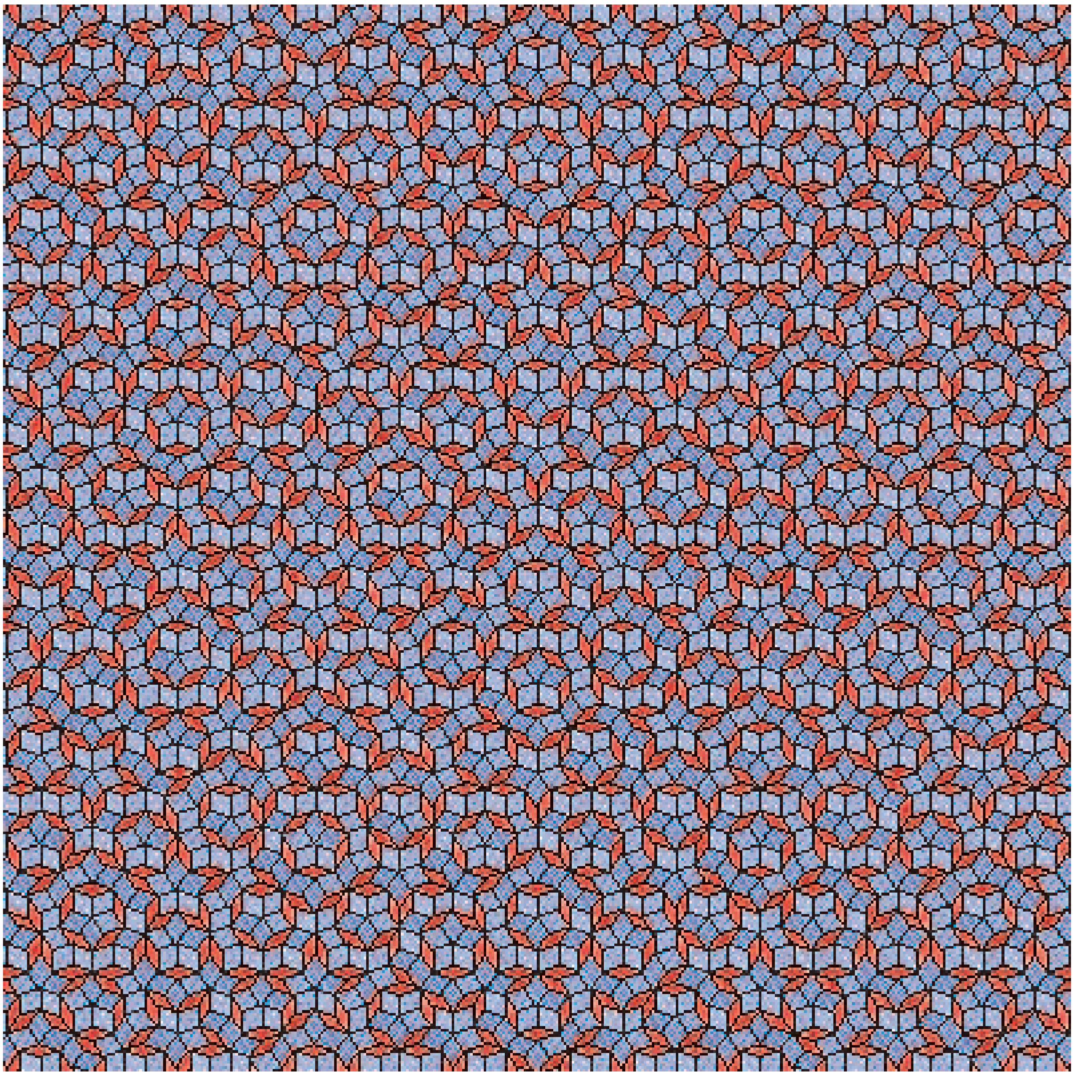

Figure 1: A Penrose tiling. Generated using Quasitiler.

# 1 General Tilings

A more general tiling construction can be obtained from the 'Penrose tiling' construction by using the same ideas on a more general space. In this section I will prove that a given subset $P$ of the integer lattice in $\mathbb{R}^n$ projected by $\pi$(an orthogonal projection) onto a m dimensional subspace $E$ generates a tiling of the subspace. This tiling has a dual that is itself a tiling (the multigrid tiling). This corresponds to the duality of the tiling and the pentagrid in the Penrose case.

I am going to assume:

- E is irrational. Thus if $E_0$ is E translated so that $0 \in E_0$ then $E_0 \cap \mathbb{Z}^n = \{0\}$.
- E is regular. Thus E does not contain any points that are integer in more than m directions. Let $M$ be the set of points in $E$ that are integer in exactly $m$ directions.

In Section 1.1 we will briefly see the idea of the construction, and establish the concepts needed for the proof.

In Section 1.2 a number of statements are proved leading to the main result in Theorem 1.7

There are a number of figures illustrating important concepts. I have kept these simple. Obviously all the subtle problems are lost in these pictures, but they are intended as an aid to visualising the concepts.

## 1.1 Preliminaries

The idea of the construction is that given the base space $\mathbb{R}^n$ and $E$ the target subspace, we consider the projection of 'edges' of the integer lattice that are entirely contained in a 'strip' of the base space. (See figure 2)

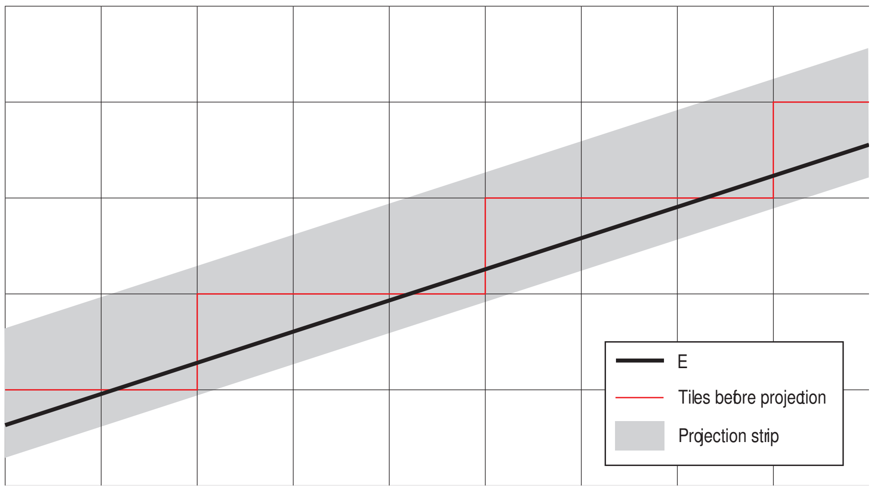


Figure 2: Here the base space is $\mathbb{R}^2$ with the integer lattice. E corresponds to the target space with the projection strip defined by translating the base point of a square along E. Any edge of the integer lattice that is entirely contained in the projection strip belongs to the 'Tiles before projection'. (These do not show up well on black and white copies)

Many concepts need to be defined. They build on each other and form the basis for the theorems, that are considered in Section 1.2.

**Concept** I define a convex polyhedron to be the intersection of a number of half-spaces ($HS_{i,\lambda} := \{y \in \mathbb{R}^n | \, y_i \geq \lambda\}$ is a closed half space). A face of a convex polyhedron is obtained by replacing a number of the half-spaces($HS_{i,\lambda}$) with the equivalent hyperplanes($H_{i,\lambda} = \{x \in \mathbb{R}^n | x_i = \lambda\}$).

**Concept** Given a convex polyhedron $C \subset \mathbb{R}^n$. We can define the tangents (Figure 3) and normals (Figure 4) at $x \in \delta C$ as follows:

$$\begin{aligned} T_x(C) &:= \{v | \exists c : [0, \epsilon] \to \mathbb{R}^n \text{ with } c(0) = x \; c'(0) = v \; c((0, \epsilon]) \subset C\} \\ &= \{v | \exists \epsilon > 0 \; x + tv \in C \; \forall t \in (0, \epsilon]\} \quad \text{(for a convex polyhedron)} \\ N_x(C) &:= \{z | \langle z, v \rangle \leq 0 \forall v \in T_x(C)\} \end{aligned}$$

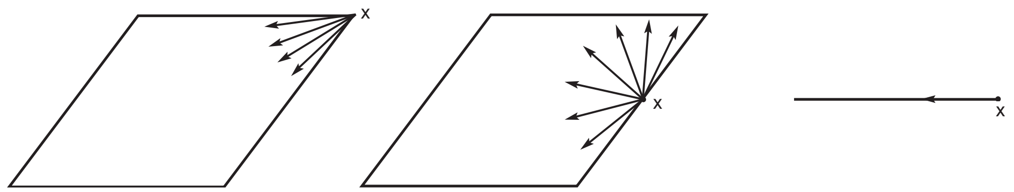


Figure 3: The set of arrows represent the tangents at the point x to various tiles C, i.e. $T_x(C) = \{v | \exists \epsilon > 0 \; x + tv \in C \; \forall t \in (0, \epsilon]\}$

I extend the definitions of tangent and normal from points to edges, faces or in general 'tiles', of the boundary. So given a convex polyhedron C (either open or closed), and t (open) a tile, where $t \subset \delta C$, then I can define $T_t(C) = T_x(C) \; x \in t$ and $N_t(C) = N_x(C) \; x \in t$. This definition is well defined since $T_x(C) = T_{x'}(C) \; \forall \; x, x' \in t$ and $N_x(C) = N_{x'}(C) \; \forall \; x, x' \in t$.

**Concept** Now to the *multigrid*. This is simply $E$ intersected with the integer lattice, but it turns out to be dual to the 'Penrose tiling'. Formally I mean:

$$E \cap \bigcup_{1 \leq i \leq n \text{ and } k \in \mathbb{Z}} H_{i,k}$$

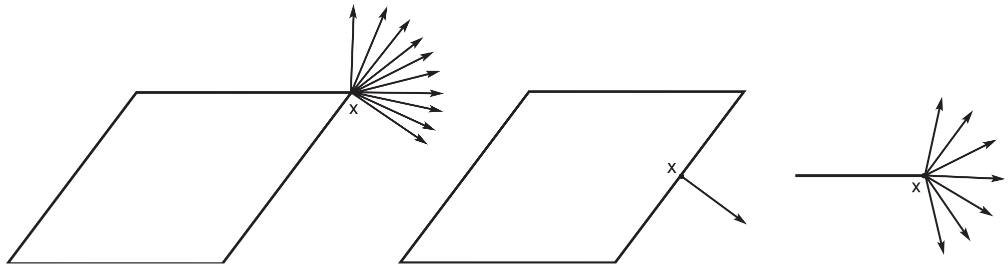


Figure 4: The set of arrows represent the normals at the point x to various tiles C, i.e. $N_x(C) = \{z | \langle z, v\rangle \leq 0 \forall v \in T_x(C)\}$

**Notation** Some notation I use given $r \in \mathbb{R}$ is:

- the floor $\lfloor r \rfloor = \max\{i \in \mathbb{Z} |\ i \leq r\}$
- the ceiling $\lceil r \rceil = \min\{i \in \mathbb{Z} |\ i \geq r\}$.

**Concept** Let $Cell_x$ be the *cell* of the multigrid which has $x$ in its interior, where $x$ is in $E$, as illustrated in figure 5. I assume a general notion of cell of any dimension. So every point in $E$ will lie in exactly one cell. To formalise this I need to also define $A_x$ which is simply the largest open k-cube defined by the integer lattice containing x. So formally assuming x is integer in the first j directions:

$$A_x = \{y \in \mathbb{R}^n |\ y_i = x_i \ 1 \leq i \leq j \ y_i \in (\lfloor x_i \rfloor, \lceil x_i \rceil) i > j\}$$

$$Cell_x = E \cap A_x$$

**Concept** Let $C$ be the real closed unit cube. I consider the closed cube which makes a lot of the definitions easier. If considering the open cube seems easier that is fine, since, from the fact that $E$ is regular and irrational, the tilings obtained are the same (see remark 1.1). So define $C_x := \{y \in \mathbb{R}^n |\ y_i \in [x_i, x_i + 1]\}$, ie the unit cube with base point $x$.

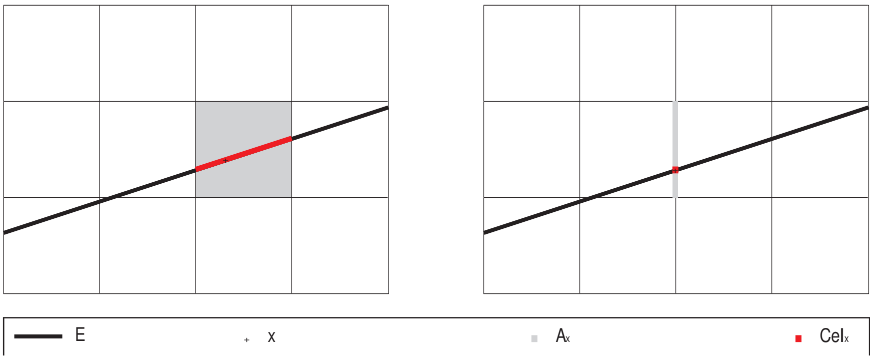


Figure 5: Illustration of multigrid components in a trivial case, for x situated on a one dimensional cell (left hand figure) and a zero dimensional cell.

**Concept** A key notion is that of the 'Projection strip' which is the subset of our base space that we will be considering. For 'Penrose tilings' the definition that is often used is $E \oplus C := \{y \in \mathbb{R}^n | \exists x \in E \; with \; y \in C_x\}$ but for our purposes this is equivalent (see remark 1.1) to $PS = \{y \in \mathbb{R}^n | \exists x \in M \; with \; y \in C_x\}$ which is a more useful definition.

Let $P$ be the set of 'Projection' integers, then $P := \mathbb{Z}^n \cap PS$. These are the points that when projected will give us the points in our 'Penrose type tilings'.

The following remark establishes why the definitions of the 'projection strip' are equivalent. Figures 6 and 7 help to give a picture of what the two strips are.

**Remark 1.1** *$P = \mathbb{Z}^n \cap (E \oplus C) = \mathbb{Z}^n \cap (E \oplus C')$, where $C'$ is the interior of $C$*

*Proof.* $P \subset \mathbb{Z}^n \cap (E \oplus C)$ is obvious since $PS \subset E \oplus C$.

For all $x \in \mathbb{Z}^n \cap (E \oplus C)$ there exists a $y \in E$ such that the cube with origin $y$ contains $x$. Since $E$ is m-dimensional we can translate $y$ in $E$ to $y'$

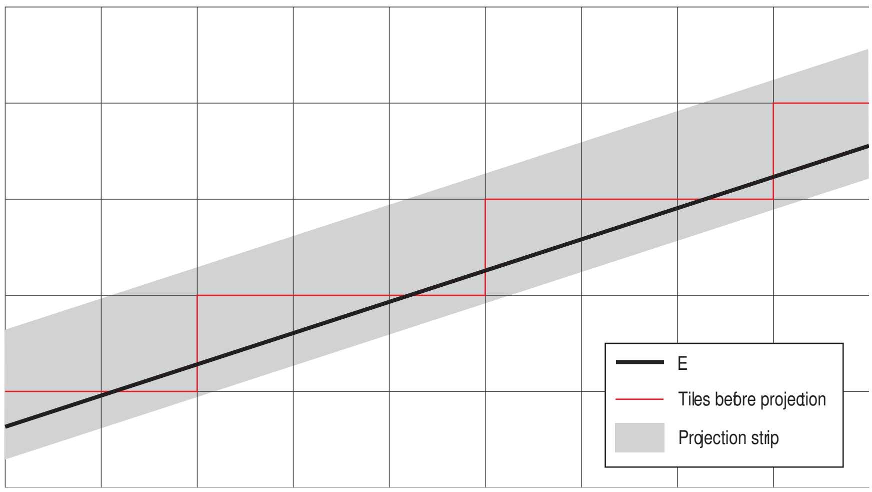


Figure 6: Projection idea with $E \oplus C := \{y \in \mathbb{R}^n | \, \exists \, x \in E \; with \; y \in C_x\}$

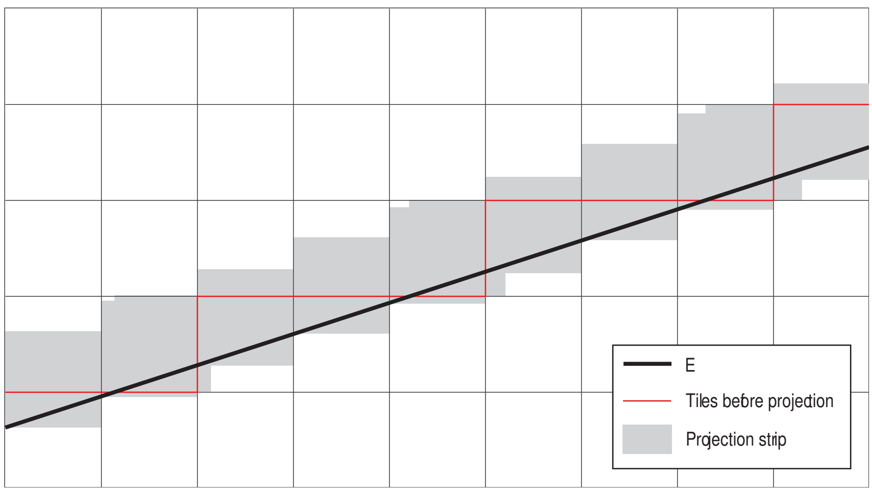


Figure 7: Projection idea with $PS = \{y \in \mathbb{R}^n | \, \exists \, x \in M \; with \; y \in C_x\}$

such that $x$ lies on a m dimensional boundary of the cube with origin $y'$. But $x_i$ is integer for all i and $x$ is 0 or 1 away from $y'$ in m directions. So $y'$ is integer in m directions, hence $y' \in M$.

$\mathbb{Z}^n \cap (E \oplus C) \supset \mathbb{Z}^n \cap (E \oplus C')$ is obvious.

Consider the projection $\pi_\perp$ of $E \oplus C$ onto $E^\perp$. Let $f$ be an open face of a $C_x$ cube ($x \in E$) such that $\dim f \geq n - m$. Since E is irrational $\pi_\perp(f)$ is an open subset of $E^\perp$. Hence can not contain any points of the boundary. Hence a point on the boundary can only lie on a face of dimension less than $n - m$. So if a boundary point y belonged to $\mathbb{Z}^n$ then for $x \in E$ such that $y \in C_x$, x must be integer in more than $n - (n - m) = m$ directions. Since E is regular, this is a contradiction. Hence $\mathbb{Z}^n \cap (E \oplus C) = \mathbb{Z}^n \cap (E \oplus C')$

*Q.E.D*

**Concept** For all $x$ in E let $I_x = C_x \cap P$ the 'integer' lattice points corresponding to x.

**Note** For x in E with j integer coordinates. $I_x$ is $2^j$ integer lattice points forming an j dimensional unit cube.

*Proof.* without loss of generality $x_1$ to $x_j$ are the integer coordinates. So for $y$ in $C_x$:

$$y_i \in \mathbb{Z} \Leftrightarrow \begin{cases} y_i = x_i \text{ or } x_i + 1 & i \leq j \\ y_i = \lceil x_i \rceil & i > j \end{cases}$$

*Q.E.D*

**Concept** Call $t_x$ the *tile* obtained by the projection onto E of the j dimensional cube ($B_x$) defined by the points of $I_x$.

Working with this concept of a tile makes it possible to keep the proof independent of the dimension. However the intuitive 'tile' is a maximum dimensional tile, which is a useful concept to have, so I define $\mathbb{T} = \{t | t = \overline{t_x} \text{ with } x \in M\}$, ie the set of maximal dimensional tiles.

**Remark 1.2** *$t_x$ is a $j$ dimensional convex polyhedron with $2^j$ vertices. And there are only finitely many different tile types. And any tile has minimal 'width' $\epsilon$ for some fixed $\epsilon > 0$. (I explain the concept of minimal width in the proof)*

*Proof.* Convexity is assured since $t_x$ is is the projection of a convex polyhedron.

The number of vertices are preserved because the projection is orthogonal to $E$ which is regular and irrational.

The finiteness follows from the fact that all tiles are projection of a j-dimensional unit cube in one of only finitely many positions up to translation.

By a tile having 'width' $\epsilon$ I mean that a j dimensional ball of radius $\epsilon$ can be fitted into the tile. Given the finiteness of the tiles a minimum width $\epsilon$ must exist.

*Q.E.D*

**Concept** We need to define the concept of neighboring m dimensional tiles. (i.e. tiles of the form $t_x$ $x \in M$.) This could be done directly but a simpler alternative, which I use, is through duality[1].

Let $x, y \in M$ then $t_x$ and $t_y$ are said to be neighbors if and only if there exists $z$ such that $Cell_z$ is a cell joining $Cell_x$ to $Cell_y$ (i.e. $\overline{Cell_z}$ contains $Cell_x$ and $Cell_y$). Then $\overline{t_x}$ and $\overline{t_y}$ have a common tile $t_z$.

**Concept** Now define $S = \bigcup \mathbb{T}$ (i.e. the disjoint union of the maximum dimensional tiles). Define also the equivalence relation $\sim$ over $S$, such that for $p, q \in S$ with $p$ in a tile $s$ and q in a tile $t$. We have $p \sim q$ if and only if $s$ and $t$ are neighbors and $\pi$ maps $p$ to $q$. This gives $\mathbb{S} = S/\sim$ and $\varrho$, the projection from $S \to \mathbb{S}$.

[1] Duality between tiles and cells will be proved later, in lemma 1.8.

**Definition 1.3** *Given $t \in \mathbb{T}$. I define*

$$U_t = \bigcup \{s \in \mathbb{T} | s \text{ and } t \text{ are neighbors } \}/\sim$$

*as the neighborhood of $t$.*

This can be extended to be the neighborhood of any p in $\mathbb{S}$ in the following way:

$$U_p = \bigcap_{t \in \mathbb{T} | p \in t} U_t$$

**Remark 1.4** *It is a direct consequence from the local tiling result (to be presented and proved later) that $U_p$ has radius at least $\epsilon$ around $p$ in every direction, where $\epsilon > 0$ is smaller than the minimum 'width' of the tiles.*

Recap, assuming $x \in E$ is integer in the first $j$ coordinates ($0 \leq j \leq m$):

$$
\begin{array}{rcl}
E & := & m \text{ dimensional regular and irrational subspace of } \mathbb{R}^n \\
M & := & \{y \in E | y \text{ is integer in m directions}\} \\
C_x & := & \{y \in \mathbb{R}^n |\, y_i \in [x_i, x_i + 1]\} \text{ (ie the unit cube with base point x)} \\
A_x & := & \{y \in \mathbb{R}^n |\, y_i = x_i \; 1 \leq i \leq j \; y_i \in (\lfloor x_i \rfloor, \lceil x_i \rceil) i > j\} \\
Cell_x & := & E \cap A_x \text{ (A cell of the multigrid)} \\
PS & := & \{y \in \mathbb{R}^n |\, \exists x \in M \; with \; y \in C_x\} \text{ (The 'projection' strip)} \\
P & := & \mathbb{Z}^n \cap PS \text{ (The 'Projection' integers)} \\
I_x & := & C_x \cap P \text{ (The integers corresponding to x)} \\
B_x & := & Span(I_x) \cap \dot{C}_x \text{ (The open cube defined by } I_x) \\
t_x & := & \pi(B_x) \text{ (The tile defined by } I_x) \\
\mathbb{T} & := & \{t | t = \overline{t_x} \text{ with } x \in M\} \\
S & := & \bigcup \mathbb{T} \\
p \sim q & \Longleftrightarrow & \pi(p) = \pi(q) \text{ and p and q belong to neighboring tiles} \\
U_t & := & \bigcup \{s \in \mathbb{T} | s \text{ and } t \text{ are neighbors } \}/\sim
\end{array}
$$

## 1.2 The Tiling

**Definition 1.5** *A tiling of a space X (*$\dim(X) = m$*) is the union of compact polyhedrons that cover X, with the property that the interiors do not intersect and when the closures intersect they intersect in an i dimensional face where* $0 \leq i < m$.

A polyhedron has not been defined above, but convex polyhedrons can be used instead without an effect on the following proofs.

**Remark 1.6** *The multigrid is a tiling.*

*Proof.* The multigrid is defined as the intersection of the unitary tiling with E. So the tiles are convex polhedrons and the tile intersections have to be subsets of tile intersections in the unitary tiling and hence valid. The covering of E follows from the fact that the unitary tiling covers $\mathbb{R}^n$.

*Q.E.D*

**Theorem 1.7**

$$\bigcup_{t \in \mathbb{T}} t \textit{ is a tiling of } E$$

.

*Proof.* The proof of this theorem requires a number of steps. The first step is to establish the duality of the multigrid and the projected tiles. Then we need to establish that locally the 'Penrose' representation is a tiling. Finally the global tiling results from the local result.

Lemma 1.8 establishes the duality.

The local result corresponds to showing that for any m dimensional tile $t$ its neighbors satisfy the definition of tiling. This implies that the neighborhood $U_t$ is an open subset of $\mathbb{R}^m$ surrounding t.

The fact that closures of tiles can only intersect in a lower dimensional tile follows from the duality of cells and tiles. If the closures of two tiles intersect, then the corresponding cells lie in the closure of a cell. By duality the tile corresponding to this cell lies in the intersection.

The fact that the interiors do not intersect follows from Claim 1.10 and the bijective correspondence between cells and tiles, since from a given tile the tangent vectors into other tiles come from the corresponding normal vectors in the dual. (See figure 8 for this correspondence)

The last result we need is that we have an open neighborhood in $\mathbb{S}$ around the closure of every tile in $S/\sim$. This results from the claim 1.10 and the bijective correspondence as follows. Given an i dimensional tile the corresponding cell has normal vectors going out from it covering $\mathbb{R}^{m-i}$. Hence there are tangent vectors going out of the tile into other tiles covering $\mathbb{R}^{m-i}$, so there is always an open neighborhood. (see figure 9)

Thus we have a local tiling and hence a tiling of $\mathbb{S}$, since $\mathbb{S}$ is defined such that the global conditions will hold.

From our projection $\pi$ which maps the tiles to $E$ we can define $\pi'$ such that the following diagram commutes. This follows because $x \sim y \Rightarrow \pi(x) = \pi(y)$.

$$\begin{tikzcd} S \arrow[r, "\pi"] \arrow[d, "\varrho"] & E \\ \mathbb{S} \arrow[ur, "\pi'"'] & \end{tikzcd}$$

From Lemma 1.11 $\pi'$ is a homeomorphism. But our tiles form a tiling on $\mathbb{S}$ and $\pi'$ maps a tile in $S/\sim$ to a tile in E. Hence we have a tiling of E.

*Q.E.D*

Theorem 1.7 has established the result. The remainder of the section completes the formal argument as has been referenced in the proof.

**Lemma 1.8** *$t_x$ and $Cell_x$ have a bijective correspondence and $\langle u, v\rangle = 0 \;\forall\; u \in T_{t_x}(t_x)$ and $v \in T_{Cell_x}(Cell_x)$ (i.e. they are perpendicular or void) for all $x \in E$.*

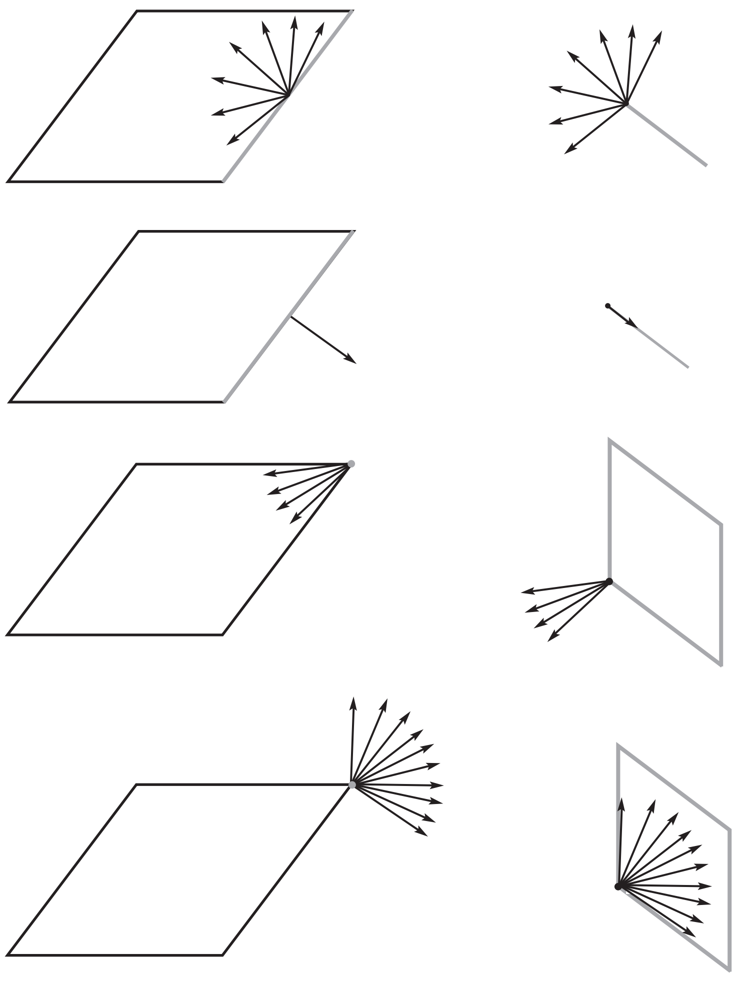

Figure 8: Each pair of figures illustrates the correspondence between tangent vectors and normal vectors in their dual.

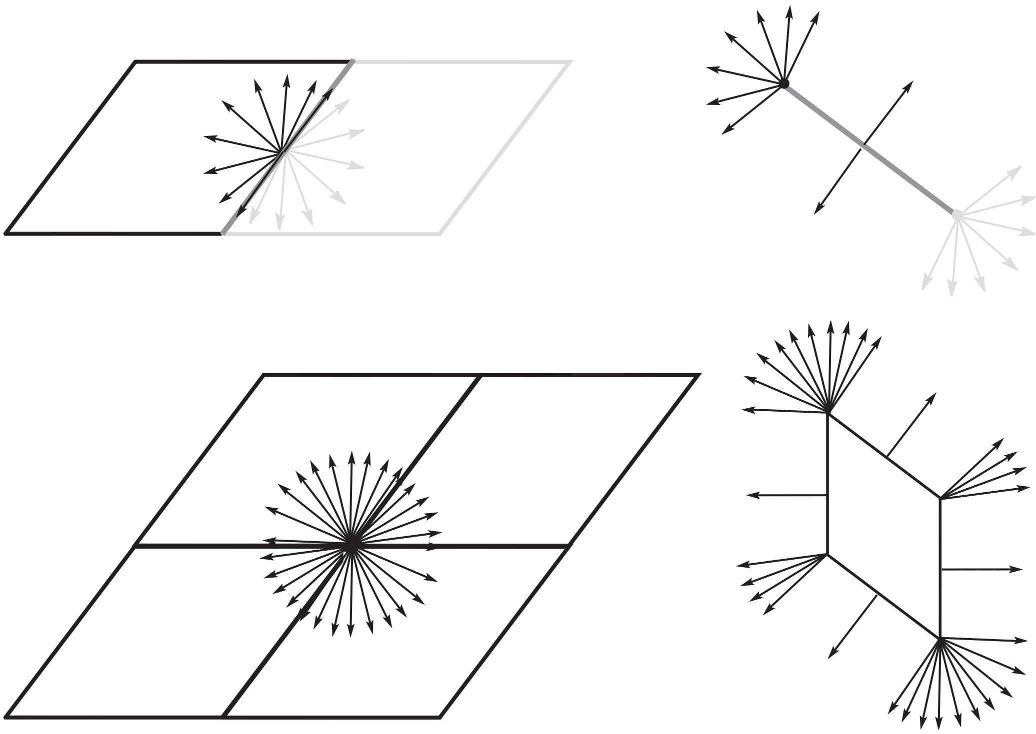

Figure 9: The right-hand figures illustrate the known fact that the normals to a cell cover the space. The left-hand figures illustrate how the the existence of an open neighborhood is hence assured by the tangent vectors.

*Proof.* We have well defined surjective functions from $x$ to both $Cell_x$ and $t_x$. Hence all that is left to show is: $Cell_x = Cell_y \iff t_x = t_y \; \forall x, y \in E$. But $Cell_x = Cell_y$ implies that x and y are both integer in j directions, and by definition $x \in Cell_x$, hence $x \in Cell_y$. So, assuming without loss of generality that x is integer in the first j directions, $x_i = y_i \; 1 \leq i \leq j$ and $\lceil x_i \rceil = \lceil y_i \rceil \; i \geq j$. Hence $I_x = I_y$ and it follows that $t_x = t_y$. Similarly $t_x = t_y$ implies that $I_x = I_y$, so by the same reasoning as above $Cell_x = Cell_y$.

For perpendicularity consider $A_x$ and $B_x$. By definition $A_x$ is fixed in the directions where $x$ is integer and $B_x$ is fixed in the directions where $x$ is not integer. Hence $\langle u', v' \rangle = 0$ for all $u' \in T_{B_x}(B_x)$ and $v' \in T_{A_x}(A_x)$. Lemma 1.9 implies $\langle u, v \rangle = 0 \; \forall u \in T_{t_x}(t_x)$ and $v \in T_{Cell_x}(Cell_x)$.

*Q.E.D*

**Lemma 1.9** *Given any $x$ in $E$ and $y$ in the closure of $Cell_x$. If $\langle u', v' \rangle \leq 0$ for all $u' \in T_{B_y}(B_x)$ and $v' \in T_{A_x}(A_y)$, then $\langle u, v \rangle \leq 0$ for all $u \in T_{t_y}(t_x)$ and $v \in T_{Cell_x}(Cell_y)$. Further if $\langle u', v' \rangle = 0 \; \forall \; u', v'$ then $\langle u, v \rangle \leq 0 \; \forall \; u, v$*

*Proof.* Given $u \in T_{t_y}(t_x)$ there exists $u' \in T_{B_y}(B_x)$ such that $u = \pi(u')$. We note first that since the projection is self adjoint we get

$$\langle u, v \rangle = \langle \pi(u'), v \rangle = \langle u', \pi(v) \rangle = \langle u', v \rangle$$

but since the multigrid is defined by inclusion we get $v \in T_{A_x}(A_y)$. Hence $\langle u', v \rangle \leq 0$ or $\langle u', v \rangle = 0$ by assumption.

*Q.E.D*

**Claim 1.10** *Given any $x$ in $E$ and $y$ in the closure of $Cell_x$ then $T_{t_x}(t_y) = \dot{N}_{Cell_y}(Cell_x)$.*

*Proof.* First consider $T_{B_x}(B_y)$ and $T_{A_y}(A_x)$. Assuming without loss of generality $x$ is integer in its first $j$ components, and $y$ in its first $l$ components with $m \geq l \geq j$.

Recall:

$$
\begin{aligned}
B_x &= \left\{ z \middle| \begin{cases} z_i \in (x_i, x_i+1) & i \leq j \\ z_i = \lceil x_i \rceil & i > j \end{cases} \right\} \\
A_x &= \left\{ z \middle| \begin{cases} z_i = x_i & i \leq j \\ z_i \in (\lfloor x_i \rfloor, \lceil x_i \rceil) & i > j \end{cases} \right\}
\end{aligned}
$$

Giving

$$
\begin{aligned}
T_{B_x}(B_y) &= \left\{ v \middle| \begin{cases} v_i \in \mathbb{R} & i \leq j \\ v_i < 0 & j < i \leq l \; y_i < x_i \\ v_i > 0 & j < i \leq l \; y_i > x_i \\ v_i = 0 & l < i \end{cases} \right\} \\
T_{A_y}(A_x) &= \left\{ v \middle| \begin{cases} v_i = 0 & i \leq j \\ v_i > 0 & j < i \leq l \; y_i < x_i \\ v_i < 0 & j < i \leq l \; y_i > x_i \\ v_i \in \mathbb{R} & l < i \end{cases} \right\}
\end{aligned}
$$

We know $A_y \perp B_y$ and $B_x \subset \bar{B}_y$ so $A_y \perp B_x$.

Take $v \in T_{B_x}(B_y)$ and $v' \in T_{A_y}(A_x)$ then:

- $v_i v'_i = 0$ For $l < i$ or $i \leq j$ since $v_i = 0 \; i > l$ and $v'_i = 0 \; i < j$.
- $v_i v'_i < 0$ For $l \geq i > j$. There are two possibilities. Either $y_i < x_i$, which implies that $v_i < 0$ and $v'_i > 0$ or $y_i > x_i$ which implies that $v_i > 0$ and $v'_i < 0$. In both cases $v_i v'_i < 0$.

These implies $\langle v, v' \rangle \leq 0$. Hence $T_{B_x}(B_y) \subset N_{A_y}(A_x)$ but $T_{B_x}(B_y)$ is open so $T_{B_x}(B_y) \subset \dot{N}_{A_y}(A_x)$.

Similarly for any $v$ such that $\langle v, v' \rangle \leq 0$ for every $v' \in T_{A_x}(A_y)$ (i.e. $v \in N_{A_y}(A_x)$) then:

- For $l < i$ we have $v'_i = 0$ hence there are no constraints on $v_i$

- For $i \leq j$ there are no constraints on $v'_i$, so $v_i = 0$
- For $l \geq i > j$ again there are two possibilities. Either $y_i < x_i$, which implies that $v'_i > 0$, giving $v_i \leq 0$. Or $y_i > x_i$, which implies that $v'_i < 0$, giving $v_i \geq 0$.

So $v \in \overline{T_{B_x}(B_y)}$. This gives us $\overline{T_{B_x}(B_y)} \supset N_{A_y}(A_x)$. Taking the interior of both sides we get $T_{B_x}(B_y) \supset \dot{N}_{A_y}(A_x)$.

So $T_{B_x}(B_y) = \dot{N}_{A_y}(A_x)$. Also $\dim(N_{A_y}(A_x)) = n - \dim(T_{A_y}(A_x))$.

But we get to $Cell_x$ from $A_x$ by intersection with E, with $\dim(T_{Cell_y}(Cell_x)) = \dim(T_{A_y}(A_x)) - (n-m)$ from the irrationality of E and the fact that $\dim(T_{A_y}(A_x)) \leq m$. And we got to $t_x$ from $B_x$ by projection onto E, with $\dim(T_{t_x}(t_y)) = \dim(T_{B_x}(B_y))$ since E is irrational and $\dim(B_x) < m$. These implies:

$$\begin{aligned}
\dim(T_{t_x}(t_y)) &= \dim(T_{B_x}(B_y)) \\
&= \dim(N_{A_y}(A_x)) \\
&= n - \dim(T_{A_y}(A_x)) \\
&= n - \dim(T_{Cell_y}(Cell_x)) - (n-m) \\
&= m - \dim(T_{Cell_y}(Cell_x)) \\
&= \dim(N_{Cell_y}(Cell_x))
\end{aligned}$$

From Lemma 1.9 we get $T_{t_y}(t_x) \subset \dot{N}_{Cell_x}(Cell_y)$. Hence from the dimensions $T_{t_y}(t_x) = \dot{N}_{Cell_x}(Cell_y)$.

*Q.E.D*

**Lemma 1.11** *$\pi'$ is a homeomorphism*

*Proof.* In Lemmas 1.12 and 1.13 I establish that $\pi'$ is surjective and a homeomorphism in an $\epsilon$ neighborhood of any point. So injectivity is all that is left to show.

Lemma 1.14 states that $\mathbb{S}$ is a flat complete Riemmanian manifold so if $(S/\sim, \pi')$ is a covering of E. Then since they are both flat $\pi'$ must be a homeomorphism.

A more general theorem could be used here to establish the covering result (see ). But a direct proof is simpler. I need to show that given any $x \in E$ there exists a neighborhood $V$ of $x$ such that $\pi'^{-1}(V)$ is a disjoint union of $V_i'$ and $\pi'|_{V_i'}$ is a homeomorphism.

For all $p \in \pi'^{-1}(x)$ from Lemma 1.12 there is a ball $B_\epsilon(p)$ of radius $\epsilon > 0$ around $p$ such that $\pi'|_{B_\epsilon(p)}$ is a homeomorphism. Since $\mathbb{S}$ has the induced metric we have that $\pi'|_{B_\epsilon(p)}$ is an isometry, hence $\pi'(B_\epsilon(p)) = B_\epsilon(x)$. So all that is left to show is that for any $p, p' \in \pi'^{-1}(x)$ we have $B_\epsilon(p) \cap B_\epsilon(p) = \emptyset$. Assume this is false, then there exists a $q \in B_\epsilon(p) \cap B_\epsilon(p)$, so $p, p' \in B_\epsilon(q)$. This is a contradiction since $\pi'(p) = \pi'(p') = x$ and $\pi'|_{B_\epsilon(p)}$ is a homeomorphism.

*Q.E.D*

**Lemma 1.12** *$\pi' : \mathbb{S} \to E$ is a homeomorphism in a neighborhood of any point (of radius $> \epsilon$ for some $\epsilon > 0$).*

*Proof.* Using the induced topology on $\mathbb{S}$ ($U' \subset \mathbb{S}$ is open $\iff$ $U = \varrho^{-1}(U')$ is open in S.) I need to show that $\pi'$ is a continuous bijection on a neighborhood of any $p \in S/\sim$, and its inverse is also continuous. The bijectivity has already been established when restricted to $U_p$ in the local result of the main theorem (1.7). From remark 1.4 considering the induced metric on $S/\sim$ the neighborhood is of radius at least $> \epsilon$ for some $\epsilon > 0$.

The next results hold for the function globally.

$\forall U \subset E$ open we have $\pi'^{-1}(U)$ open in $\mathbb{S}$ $\iff$ $\varrho^{-1}\pi'{-}1(U) = \pi^{-1}(U)$ is open which the continuity of $\pi$.

$\forall U \subset \mathbb{S}$ open (i.e. $\varrho^{-1}(U)$ open) we have $(\pi'^{-1})^{-1}(U) = \pi'(U) = \pi' \circ \varrho(\varrho^{-1}U) = \pi(\varrho^{-1}U)$ which is open since $\pi$ is an open function.

*Q.E.D*

**Lemma 1.13** *$\pi' : \mathbb{S} \to E$ is surjective.*

*Proof.* $\pi'(\mathbb{S})$ is open since $\pi'$ is an open mapping (projections are open mappings).

Also $\pi(S)$ is closed since $S$ is the union of closed tiles (of radius at least $\epsilon > 0$) and $\pi$ is an isometry on any tile. But $\pi = \pi' \circ \varrho$ hence $\pi' \circ \varrho(S)$ is closed and since $\varrho$ is surjective $\pi'(\mathbb{S})$ is closed.

*Q.E.D*

**Lemma 1.14** $\mathbb{S}$ *is a complete $m$ dimensional flat Riemannian manifold*

*Proof.* $\forall\, p \in S \;\exists\, x \in M$ such that $p \in \overline{t_x}$. But we now know[2] that p lies in the interior of $U_x$ so we have an open neighborhood around any point that is homeomorphic to $\mathbb{R}^m$. And any Cauchy sequence will eventually be contained in one such neighborhood, so will converge. We still need to show that $\mathbb{S}$ is Hausdorf, i.e. for all $p, q \in S$ there exist open neighborhoods around them that don't intersect. Given $p, q \in S$ if there exists a tile $t$ such that $p, q \in \overline{U_t}$ then the Hausdorf condition is satisfied since $\overline{U_t}$ is Hausdorf (lemma 1.12 has established that $\overline{U_t}$ is homeomorphic to a closed subset of $\mathbb{R}^m$). If no such neighborhood exists then take tiles $s, t$ such that $p \in \bar{s}$, $q \in \bar{t}$. If $U_s \cap U_t \neq \emptyset$ then take any tile $t'$ lying in the intersection and both $s, t$ must be neighbors so both are contained in $\overline{U_t}$ hence $p, q$ lie in $\overline{U_t}$. But we assumed no such neighborhood exists. Hence $U_s \cap U_t = \emptyset$ and we have satisfied the Hausdorf definition. Flatness follows since the metric is induced from E which has a flat metric.

*Q.E.D*

[2]This is an important point. It follows from the the local tiling result

# 2 General Inflation and Deflation property

The aim of this section is to extend the generalisation of the 'Penrose' tiling informally[3]. In the previous chapter the 'Penrose' tiling construction was generalised, but the construction is not enough to get the special properties. In the 'Penrose' tiling the special properties can be explained by the inflation/deflation property.

The first aim of this chapter is to try and decompose the ideas of inflation/deflation:

- Section 2.1 presents the idea of splitting the space and the 'Penrose' example will be developed.
- Section 2.2 looks into the inflation deflation property itself.
- Section 2.3 explores the idea of acting a group on some of the directions that will be projected.

The second aim will be to introduce ideas that have not been explored fully:

- Section 2.4 explores examples and ideas for the case where $\dim E = 2$.
- Section 2.5 mentions cases with $\dim E > 2$.

## 2.1 Splitting the space

Starting with the base space $B = \mathbb{R}^n$, we need an automorphism $A$ that induces a natural splitting: $B \equiv E_0 \oplus E_1 \oplus E^{\perp}$. Each of these subspaces will have a different role in the construction.

[3]The informal nature of some of this section may come as a stark contrast to the formal approach taken in the previous section. The reason for this is that the aims of the two sections are drastically different. In the first the aim is to present and prove a general result, while the second is more exploratory and the aim is to present ideas in a readable way.

From the general tiling result we know that $E$ (the m dimensional plane onto which we project) must be irrational and regular. $E$ will be defined by $E = E_1 + v$ for some $v \in E^{\perp}$, hence $E_1$ must be irrational and $v$ must be chosen to satisfy the regularity condition.

One of the areas I don't understand fully is $E^{\perp}$. In the 'penrose' case it is interchangeable with $E_1$. I think that this could be generalised to:

$$E^{\perp} \equiv \oplus_1^k E_1$$

for some given k. This is an area that needs further thought, especially the link with the hyperplanes along the $E_0$ direction, which will be explained briefly later.

The following proposition establishes splitting in the 'Penrose' case.

**Proposition 2.1** *We can get a natural splitting of* $\mathbb{R}^5$ *by applying an appropriate automorphism.*

*Proof.* Consider $A : \mathbb{R}^5 \to \mathbb{R}^5$ then linear map such that $A(e_i) = e_{i+1}$ for $i$ =1 to 5 mod 5. I.E

$$A \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} e \\ a \\ b \\ c \\ d \end{pmatrix} \quad \forall \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} \in \mathbb{R}^5$$

Which gives:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

For an eigenvalue $\lambda$

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & 0 & 0 & 1 \\ 1 & -\lambda & 0 & 0 & 0 \\ 0 & 1 & -\lambda & 0 & 0 \\ 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1 & -\lambda \end{pmatrix} = 1 - \lambda^5$$

ie:

$$\lambda \in \{1, e^{\frac{2\pi i}{5}}, e^{\frac{4\pi i}{5}}, e^{\frac{6\pi i}{5}}, e^{\frac{8\pi i}{5}}\} \iff \lambda \in \{1, \epsilon, \epsilon^2, \bar{\epsilon}^2, \bar{\epsilon}\} \; with \; \epsilon = e^{\frac{2\pi i}{5}}$$

Consider a vector $w_\lambda = \begin{pmatrix} 1 \\ \bar{\lambda} \\ \bar{\lambda}^2 \\ \bar{\lambda}^3 \\ \bar{\lambda}^4 \end{pmatrix}$ then $Aw_\lambda = \begin{pmatrix} \bar{\lambda}^4 \\ 1 \\ \bar{\lambda} \\ \bar{\lambda}^2 \\ \bar{\lambda}^3 \end{pmatrix} = \lambda w_\lambda$. From the equalities $\lambda^5 = 1$ and $\bar{\lambda}^4 = \lambda$ which hold for all $\lambda$. Define $E_0 := \mathbb{R}.\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $E_1 := Span^{\mathbb{R}}(w_\epsilon, w_{\bar{\epsilon}})$ and $E_2 := Span^{\mathbb{R}}(w_\epsilon^2, w_{\bar{\epsilon}}^2)$. Then:

$$\mathbb{R}^5 = E_0 \oplus E_1 \oplus E_2$$

with A mapping $E_i$ to itself for each i.

*Q.E.D*

## 2.2 Inflation. Deflation.

The main idea that makes the 'Penrose' tiling so special, is the fact that if you paint every tiling in the same special way than the obtained pattern is also a 'Penrose' tiling but not the same one. This is called deflation. The inverse function is inflation, figure 10. I think in general the inverse can be the composition of multiple functions, this observation comes from considering the 7-dimensional case seen in section 2.4. [4]

In the 'Penrose' case the functions are:

$$S(e_i) := e_{(i+1 \mod 5)} + e_{(i-1 \mod 5)}$$

[4] In section 2.4 I actually turn this statement around since in the 7-dimensional case it looks easier to consider the inflation and the take the inverse(deflation) to be the composition of multiple functions.

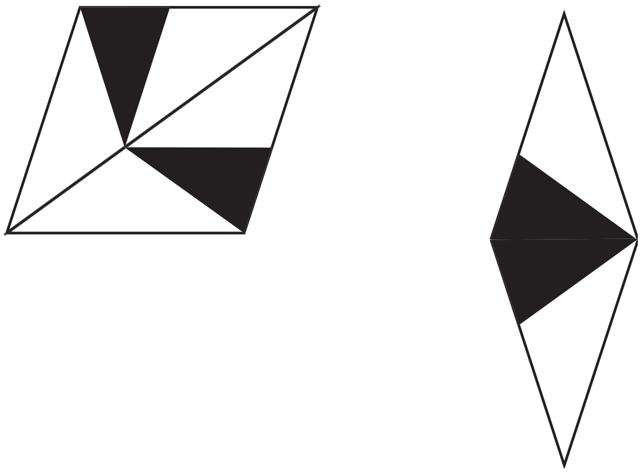

Figure 10: Tiles with subdivision

$$T(e_i) := e_{(i+2 \mod 5)} + e_{(i-2 \mod 5)}$$

$$S|_{E_1} = (\frac{-1+\sqrt{5}}{2})Id$$

$$T|_{E_1} = (\frac{-1-\sqrt{5}}{2})Id$$

Hence $(S \circ T)|_{E_1} = -Id$. In fact this also holds on $E^{\perp}$ in the 'Penrose' case.

The map S preserves the splitting. When restricted to $E_1 \oplus E^{\perp}$ it can be simply seen as shrinking in the $E_1$ direction and expanding in the $E^{\perp}$ direction(s), i.e. $\lambda$Id on each irreducible subspace of $E_1 \oplus E^{\perp}$, with $|\lambda| > 1$ on subspaces of $E^{\perp}$ and $|\lambda| < 1$ on $E_1$. The informal reason is that we want the image of the projection strip to contain the projection strip of the image of E ($E'$). (see figure 11)

What is especially important is that the images of the integer points in the projection strip of $E'$, after projection, are hit by the images of integer points in the projection strip of $E$, after projection composed with S, i.e.

$$\pi'(\cup_{x \in M'} I_x) \subset \pi'(S(\cup_{x \in M} I_x))$$

Given the following notation $I_x$ and M as defined in chapter 1, $M' := \{y \in$

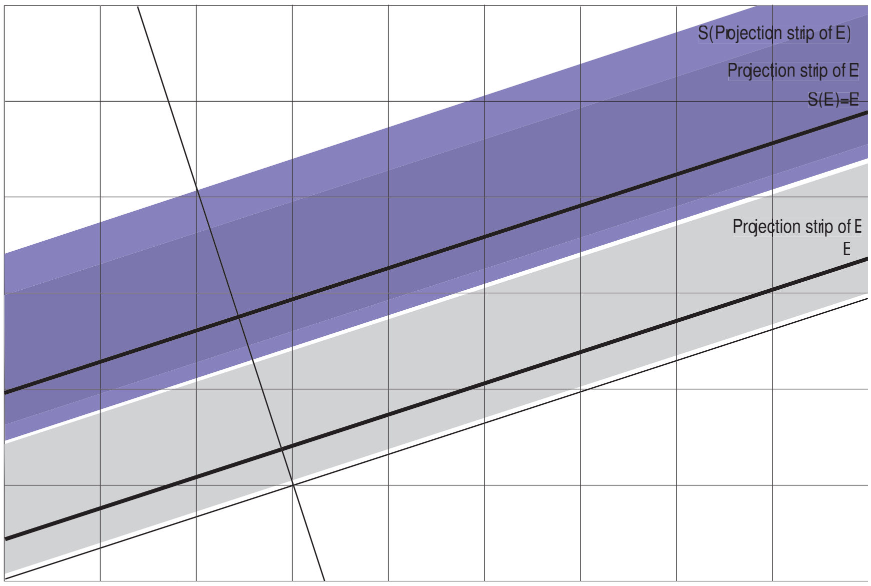


Figure 11: The action of $S$ on $E$ gives $E'$ and see here both the projection strip of $E'$ and the action of $S$ on the projection strip of $E$

$E|y$ is integer in m directions$\}$ and $\pi'(x)$ is the projection of the projection strip of $E'$ onto $E'$.

What the last statement implies is that the corner points of every tile are the corner points of some tile in the subdivision tiling. What we haven't got is the uniqueness of the subdivision of a tile. In the 'Penrose' case this follows from direct exploration that there is a unique subdivision of the tiles but in general finiteness of subdivisions might be enough and that follows directly in the same way that finiteness of the tiles follows.

In section 2.4 the 'Penrose' case will be explored further along with the other possible 'Penrose type tilings' of $\mathbb{R}^2$.

## 2.3 Working on a 'cylinder'

In the previous section we 'forgot' about $E_0$. In the 'Penrose' case one way to rectify this is to consider the set of hyperplanes (or in general subspaces) orthogonal to $E_0$ containing integer points. These can be seen as levels away from the origin along the vector $a := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. There are four levels that will intersect the original projection strip. Call $H_i$ the hyperplane through the point $\frac{i}{5}a$ for $i = 1, 2, 3, 4$.

The problem is that S will map the hyperplane $H_i$ to $H_{2i}$ so nothing maps to $H_{odd}$. The best way to see that this is no problem is to apply a group action $\mathbb{Z}$ in the $E_0$ direction identifying points which are $\alpha a$ for $\alpha$ in $\mathbb{Z}$. This means working on the infinite cylinder $\mathbb{R}^5/\mathbb{Z}$ and not on the base space $\mathbb{R}^5$.

It is clear that this does not affect the tiling result since no points in the original projection strip are identified. It also follows that the splitting is still valid and correct. This makes the mapping S applied to the set $\{H_1, H_2, H_3, H_4\}$ a bijection onto itself even though S wraps around the cylinder twice.

I think this can be applied more generally but formalising it in general requires more thought. On of the main difficult arises when $\dim E > 2$. This will be mentioned briefly in section 2.5.

## 2.4 Ideas on examples with $\dim E = 2$

A generalisation that follows the 'Penrose' construction very closely might be the only type of construction that will satisfy the inflation-deflation idea with $\dim E = 2$.

What I mean by this is that given a base space $\mathbb{R}^n$ then the splitting is $E_0 \oplus E_1 \oplus \cdots \oplus E_k$, with $E_0$ the one dimensional subspace defined by the vector $a := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and the $E_i$'s are all equivalent. Which means $k = \frac{n-1}{2}$ giving n odd and strictly bigger than five, since both $E_1$ and $E_2$ have to exist to satisfy the construction.

Also given an appropriate basis, ignoring the $E_0$ component again, and taking $j \in \{1, \ldots, k\}$ we can define:

$$S_j(e_i) := e_{(j+1 \mod n)} + e_{(j-1 \mod n)}$$

$$S_j|_{E_1} = \lambda_j Id$$

It should follow that if the above equation holds then the following equation also holds given the appropriate basis. But I have not checked the formal necessary conditions.

$$S_j|_{E_j} = \lambda_1 Id$$

These three equations specify exactly how $S$ acts on all irreducible subspaces.

Something which I think follows directly from this construction is that for $n \geq 9$ we will not get the necessary 'shrinking' on $E_1$ and 'expansion' on all irreducible subspaces of $E^{\perp}$. Hence the only 2 possible constructions are the 'Penrose' case $n = 5$ and $n = 7$. There are a number of interesting differences between the two cases. Notably:

- in the five dimensional case:

  $$S_2 \circ S_1(e_1) = e_2 + e_3 + e_4 + e_5 = a - e_1$$

  So ignoring $E_0$ gives $S_1^{-1} = -S_2$.

- in the seven dimensional case:

  $$S_3 \circ S_2 \circ S_1(e_1) = 2e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 = a + e_1$$

  So ignoring $E_0$ gives $S_1^{-1} = S_3 \circ S_2$.

In the 'Penrose' case the natural way to proceed was to define the deflation as the $S_j$ where $|\lambda_j| < 1$, and then define inflation as the inverse. In the 7 dimensional case deflation is not well defined in this way however defining inflation as the $S_j$ where $|\lambda_j| > 1$ and then considering the inverse of that to be deflation[5] is well defined.

Also the subdivision of the tile in the 'Penrose' case is very easy to find because it follows from the golden ratio, and is obvious looking at figure 10. The key in the 'Penrose' case is the different possible one dimensional tiles(edges). There are only two, the first being the long diagonal of the 'fat tile' and the second being an edge plus the short diagonal of the thin tile. The equality of these two lengths corresponds to the golden ratio.

There is no simple equivalent rule for the seven dimensional case. However if we are to get a subtiling I believe there must be a formula[6] which gives different one dimensional tiles in terms of the deflated edges and diagonals. It may be that multiple deflations are needed to find such a formula, or that the notion of deflation is chosen to satisfy such a formula. I believe that the latter approach will yield better results.

The last comments I want to add in this section concern the three figures 12, 13, and 14. I find it surprising looking at these three (all created in Quasitiler with the equivalent zooming) that supposedly the 9-dimensional case is fundamental different from the 7-dimensional case, which brings me to the following query. I wonder whether a possible construction for this 9-dimensional inflation deflation is in some way to consider a similar but slightly more complicated inflation-deflation on a 4 dimensional subspace and then consider a special 2 dimensional subset.

[5] An important thing to check would be that this new deflation satisfies the idea stated in section 2.3. Which the old on did not.

[6] This would corresponds in some sense to the condition in section 2.3 that integer points in the projection strip of $E'$ are hit, ignoring $E_0$.

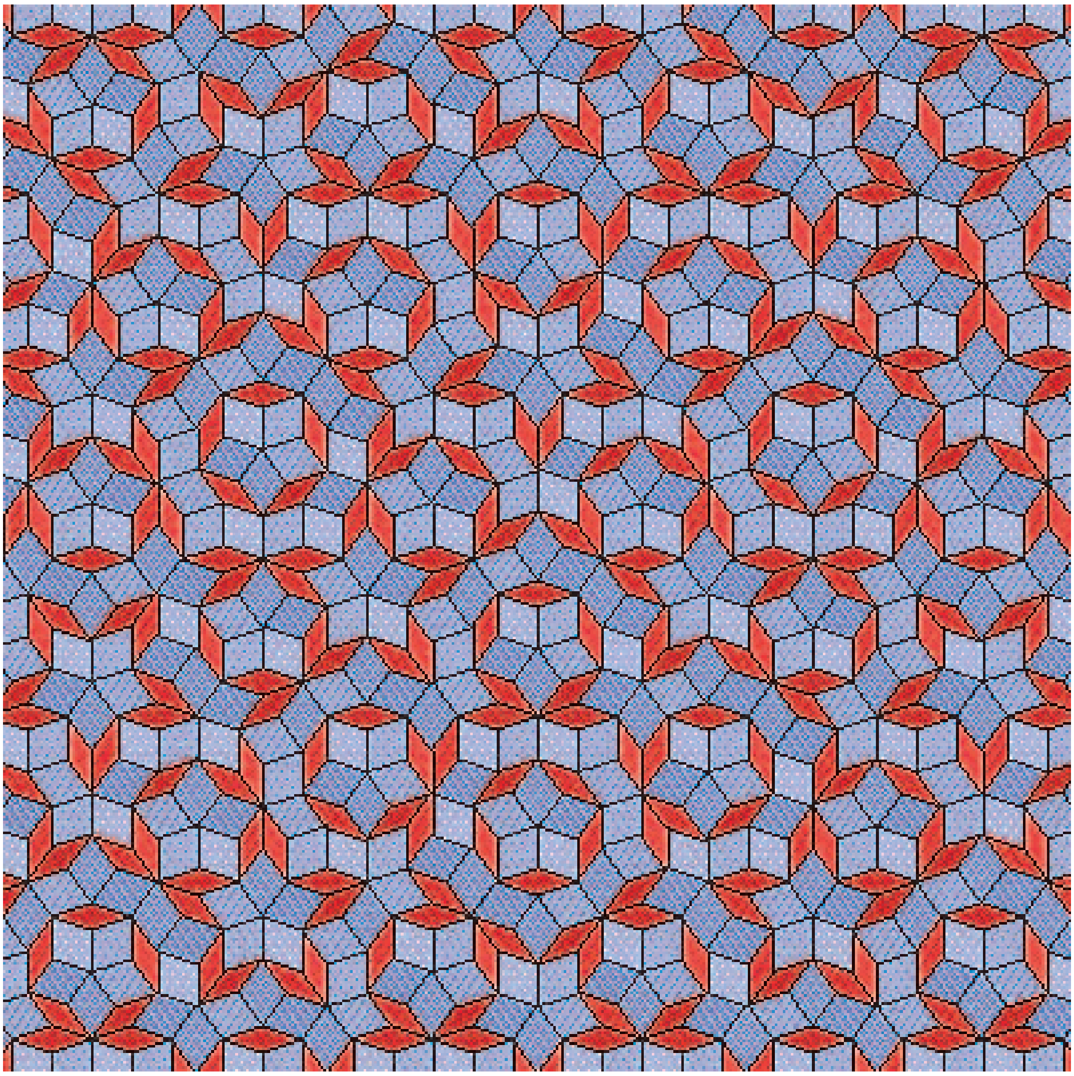

Figure 12: A Penrose tiling. Generated using Quasitiler.

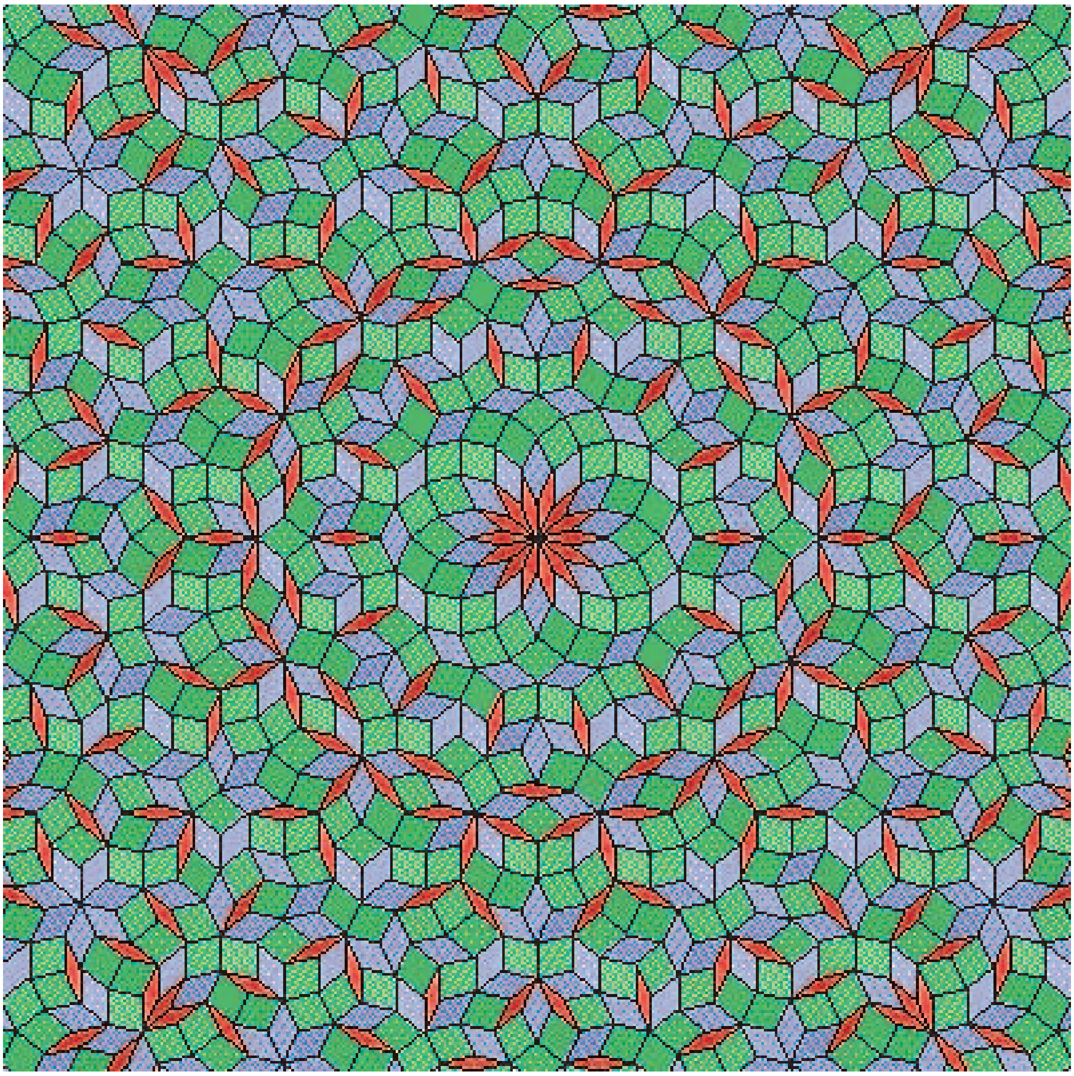

Figure 13: A tiling starting from $\mathbb{R}^7$. Generated using Quasitiler.

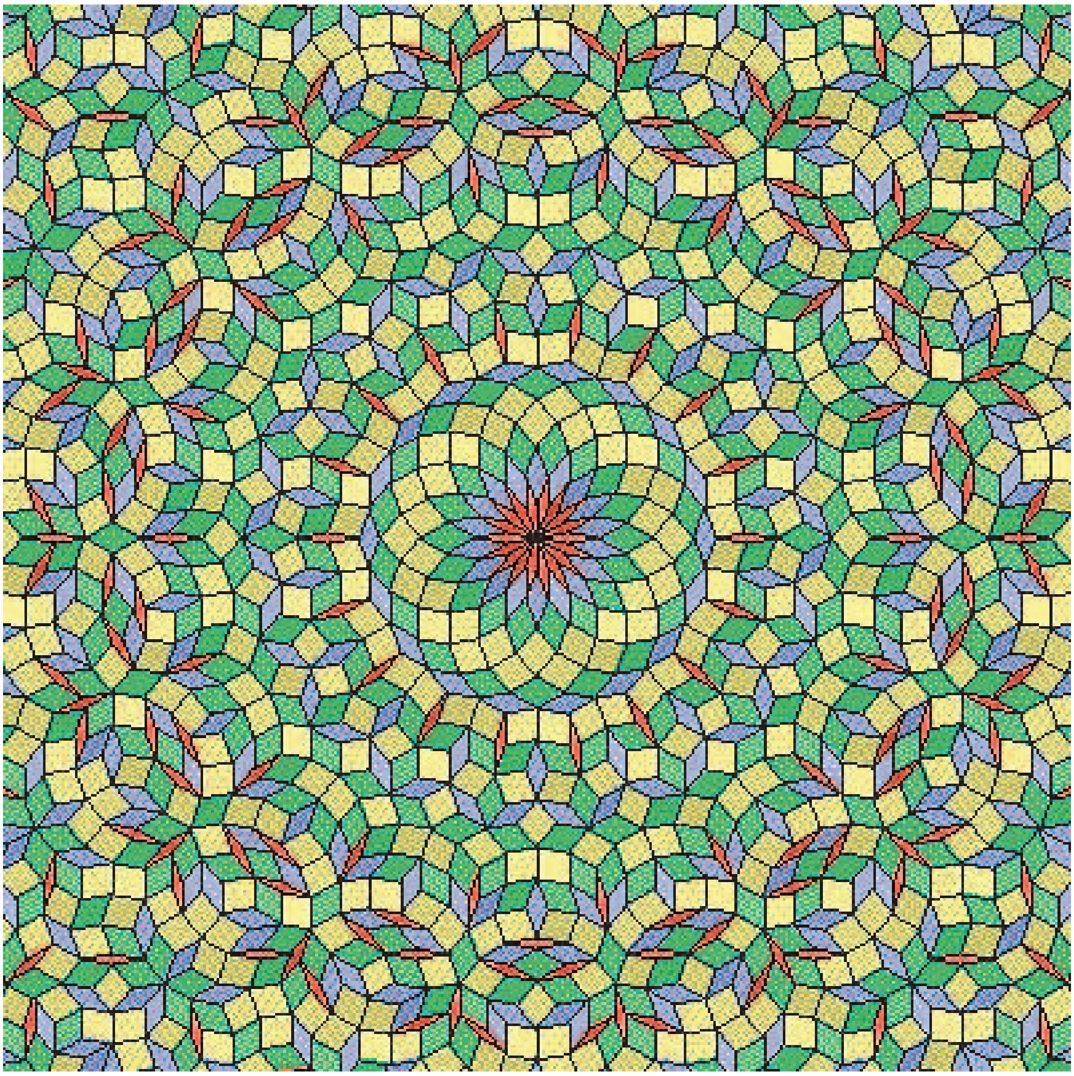

Figure 14: A tiling starting from $\mathbb{R}^9$. Generated using Quasitiler.

## 2.5 Ideas on examples with $\dim E > 2$

A large proportion of the time spent on this project was on a specific $\dim E = 3$ example. But has not yet yielded any real result. Here I merely mention some ideas for further work.

Continuing from my last comment in the previous chapter, and the construction mentioned there, it would be interesting to set $E \equiv E_i \oplus E_j$. What I mean is to look at the $k = 4$ dimensional subset obtained by considering 2 irreducible subsets. This would imply a non uniform deflation over the space. Then using the same sort of argument as in the previous section it would only be interesting for $n = 9$ or $n = 11$. And if this were interesting it would be easy to generalise to any even $k$. Wether the concept remains interesting when there is non uniform deflation is another question since some of the 'nice' properties will be lost.

For the more general approach I just mention three problems that are still unresolved.

- $\dim E > 1$. This makes the 'cylinder' approach more difficult it might still be possible using a different group action or by solving the problem of covering the integers directly.
- Orientation problems. In the example mentioned earlier this was the killing factor which kept coming back.
- No obvious inverse function. Again this was something I was always uncomfortable with. Even though it is not necessary in the construction of either the inflation or the deflation, I do not understand how interesting results can be obtained when only one can be defined without the other.

# Conclusion

The main part of this thesis is the 'tiling' result, presented in part 1. It states that when the 'Penrose type' construction is followed, then the resulting 'pattern' is a tiling.

In the 'Penrose' case this is an intuitive result, but even then the proof is not obvious. My initial aim was to prove the 'Penrose' case in a sufficiently general way that the same method could be used for other specific cases. It became apparent that a general formulation would require merely a more formal language and the avoidance of case studies in the proof.

The first part of this thesis identifies the 'Penrose type' construction as giving a tiling. This is a necessary base in the investigation of 'Penrose type tilings'. A formal statement of the tiling theorem that is independent of the proof, and an informal description of the general construction may make the second part more accessible by enabling those interested in 'Penrose type tilings' to skip the details of this proof.

The second part of the thesis is exploratory in nature, but the emphasis has been kept on taking a general standpoint. An alternative would have been to follow the 'Penrose case' through fully, lifting the argument to other specific examples. As with the first section, this was my initial approach and would still be an interesting exercise. I hope that some insight can be gained from my more general observations.

I recommend investigating the tiling of the plane using the rhombuses with angles that are multiple of $\frac{2\pi}{7}$ as the most important single case. Using the inflation and deflation described in section 2.4, it is possible that this case is relatively straightforward to follow through.

In Section 2.5 I briefly considered higher-dimensional cases. Most of my observations resulted from an exploration of a specific three dimensional tiling. Orientation was the main problem in that case and I suggest that this would be a key aspect to an understanding of this case.

If these specific cases are worthy of study then I contend that it is even more important to step back and look at the problem more generally. This was my aim in the second part. In doing so I hope to have raised some issues that need to be addressed to make a 'Penrose type tiling' well defined.

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