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WILLMORE SURFACES AND MOEBIUS GEOMETRY

The following notes grew out of discussions between J.Eschenburg, U.Pinkall and K.Voss about R.Bryants work on Willmore surfaces [Br]. In particular, the computations of ch.3 are due to Voss. The main theorem of Bryant says that Willmore spheres are Moebius transforms of certain minimal surfaces in $\mathbb{R}^3$. The proof has two sources: the theory of conformal area in Moebius geometry developed by Thomsen and Blaschke in the 20's, and the study of minimal 2-spheres in spaces of constant curvature by Calabi, Chern and others. Our aim is to exhibit these sources more clearly than in the original paper which makes the proof more transparent. We wish to thank G.Metzger who helped us understanding the Moebius geometry.

Contents.

1. The Willmore Functional

Throughout this paper, let M be a surface and $x : M \to \mathbb{R}^3$ an immersion. T.Willmore [W] proposed to consider the functional

$$W(x) = \int_M (H^2 - K)dA = \frac{1}{4} \int_M (k_1 - k_2)^2 \, dA$$

where K and H are the Gauß and the mean curvature, k_1, k_2 the principal curvatures and dA the area element of the induced metric on M. For compact surfaces, W is a measure for the total mean curvature since the Gauß-Bonnet integral $\int K dA$ does not depend on x.

It is a remarkable fact (cf. Appendix) that the functional W is invariant under conformal diffeomorphisms of the target space. In fact, by Liouville's theorem, the conformal diffeomorphisms of open subsets of $\mathbb{R}^3$ do also preserve the set of spheres and planes of $\mathbb{R}^3$ and form the group Moeb(3) of the Moebius transformations of $\mathbb{R}^3 \cup \{\infty\}$ (cf. [Sp]). For any dimension n, the group Moeb(n) is generated by isometries, homotheties and some inversion (reflection at some sphere).

<u>Lemma 1.</u> *Let S_1, $S_2 \subset \mathbb{R}^n$ be (n-1)-spheres which touch each other at some point p, i.e. they have a common normal vector n at p. Let $g \in$ Moeb(n). Then the mean curvatures k_1 of S_1 with respect to n and k_1' of $g(S_1)$ with respect to $dg_p(n)$ satisfy*

$$\lambda(p) \cdot |k_1' - k_2'| = |k_1 - k_2|$$

where λ is the metric delatation factor of g.

<u>Proof.</u> The statement is true for isometries and homotheties. We have to show it for one inversion. But there exists exactly one sphere S touching S_1 and S_2 at p such that reflection at S interchanges S_1 and S_2. Since $p \in S$, we have $\lambda(p) = 1$ for this inversion and so the formula holds trivially.

<u>Remark.</u> This sphere S is called the *central sphere* of S_1 and S_2. Its mean curvature with respect to n is $k = \frac{1}{2}(k_1 + k_2)$.

<u>Proposition 1.</u> *For any* $g \in \mathrm{Moeb}(3)$ *we have*

$$W(g \circ x) = W(x) .$$

<u>Proof.</u> For any $p \in M$, consider the bunch of spheres $S(k)$ having first order contact with $x(M)$ at $x(p)$, where the parameter k denotes the mean curvature of the corresponding sphere $S(k)$. In particular, $S(0)$ is the tangent plane at $x(p)$. For large $|k|$, the sphere $S(k)$ lies on one side of $x(U)$ for some neighborhood U of p. The supremum and the infimum of these k-values are the principal curvatures k_1 and k_2 of the surface x at p. This characterization of the principal curvatures shows that the Moebius transformation g maps the corresponding spheres $S(k_1)$ and $S(k_2)$, the so called *principal curvature spheres of* x *at* p, onto the principal curvature spheres of $g \circ x$ at p. Thus the proposition follows from Lemma 1.

Since the conformal geometry of $\mathbb{R}^3 \cup \{\infty\}$ is mapped to that of the 3-sphere S^3 by any stereographic projection $\phi : \mathbb{R}^3 \to S^3$, we also consider the analogue functional for immersions x' :

$M \to S^3$, namely

$$W'(x') = \tfrac{1}{4}\int_M (k_1' - k_2')^2 \, dA' = \int_M (1 + H'^2 - K') \, dA'$$

where the quantities with ' always refer to the immersion x'. The euclidean and the spherical curvatures are related as follows:

<u>Lemma 2:</u> *Let* $S' \subset S^n$ *be some* $(n-1)$*-sphere with spherical mean curvature* k'. *Then* k' *is the euclidean mean curvature of the sphere* $\hat{S}$ *in* $\mathbb{R}^{n+1}$ *which intersects* S^n *orthogonally along* S'.

This can be seen without computation in the case $n = 2$ by looking at the tangent cone C which touches S^n along S'.

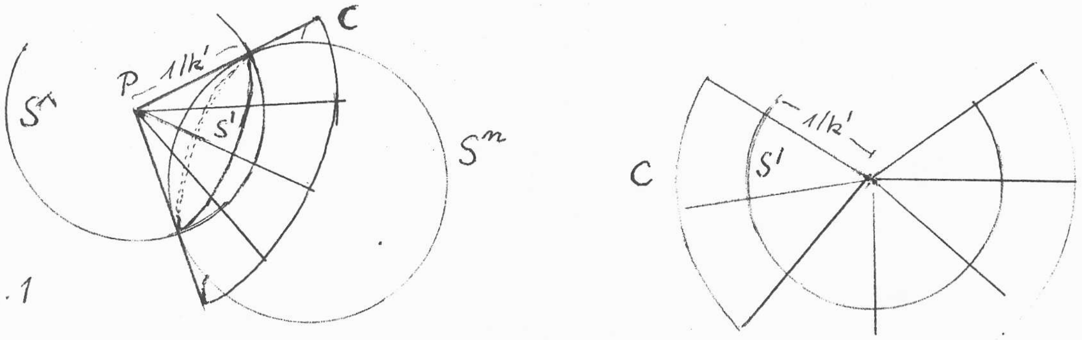

Fig. 1

To see the relation between W and W', we observe that a stereographic projection $\phi : \mathbb{R}^3 \to S^3$ with $\phi(\infty) = N \in S^3$ is the restriction to $\mathbb{R}^3 \subset \mathbb{R}^4$ of the inversion Φ at the 3-sphere $S_0 \subset \mathbb{R}^4$ centered at N and intersecting S^3 along $S^3 \cap \mathbb{R}^3$.

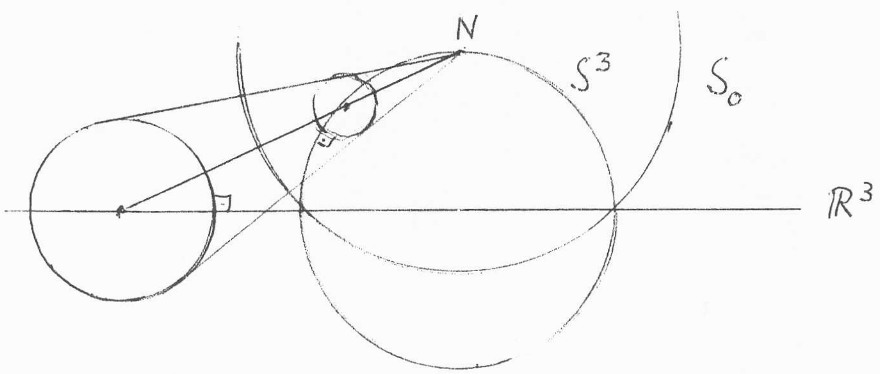

Fig. 2

As above, we see that ϕ maps the principal curvature spheres $S(k_i)$ of some immersion $x : M \to \mathbb{R}^3$ to the principal curvature spheres $S'(k_i')$ of $x' = \phi \circ x$. On the other hand, if we enlarge the 2-spheres $S(k_i) \subset \mathbb{R}^3 \subset \mathbb{R}^4$ to 3-spheres with the same center and radius, than Φ maps these onto those 3-spheres which intersect S^3 orthogonally along $S'(k_i')$. Thus from Lemma 2 and Lemma 1 (for $n = 4$) we get

<u>Proposition 2.</u> *For any stereographic projection* $\phi : \mathbb{R}^3 \to S^3$ *and any immersion* $x : M \to \mathbb{R}^3$ *we have*

$$W(x) = W'(\phi \circ x) .$$

<u>Example.</u> Consider the immersion $x' : S^1 \times S^1 \to S^3 \subset \mathbb{C}^2$,

$$x'(\sigma,\tau) = (c\cdot\sigma, s\cdot\tau)$$

where $c = \cos\alpha$, $s = \sin\alpha$ for some constant $\alpha \in (0,\pi/2)$.

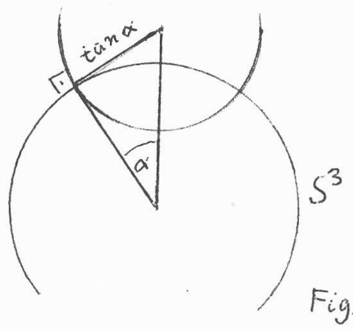

Fig.3

This arises as the enveloping surface of all 2-spheres in S^3 with spherical radius α and center on the great circle $S_1^1 = S^3 \cap (\mathbb{C} \times 0)$. These are orthogonal intersections of S^3 with 3-spheres of radius $\tan\alpha$. Likewise, x' is the enveloping surface of the spheres of spherical radius $\pi/2 - \alpha$ with center on $S_2^1 = S^3 \cap (0 \times \mathbb{C})$ which are orthogonal intersections of S^3 with 3-spheres of radius $\cot\alpha$. Since the orientations of the latter spheres are opposite, we get

$$4\, W'(x') = (\cot\alpha + \tan\alpha)^2 (2\pi\cos\alpha)(2\pi\sin\alpha)$$
$$= 2\pi^2/(\sin 2\alpha) \geq 2\pi^2$$

with equality exactly for $\alpha = \pi/4$ (Clifford torus). These surfaces correspond to the tori of revolution under a stereographic projection ϕ with $\phi(\infty) \in S_2^1$. Thus for tori of revolution we get $W \geq 2\pi^2$ with equality iff the ratio between large and small radius is $\sqrt{2}$ which corresponds to the Clifford torus.

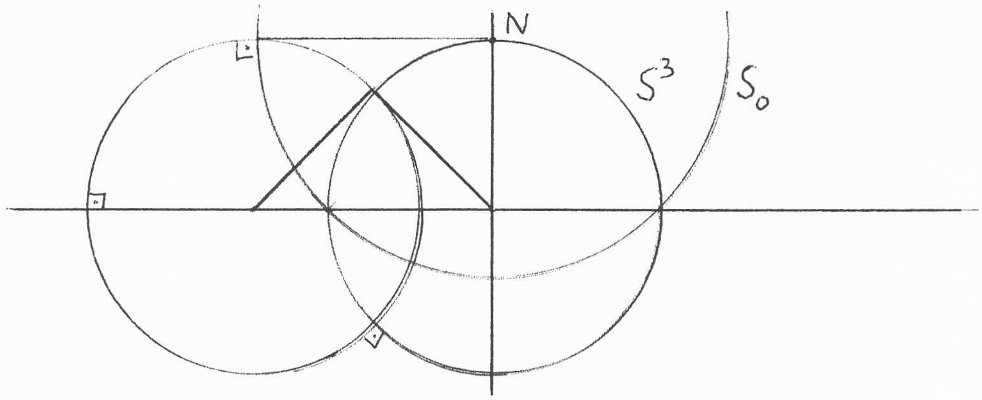

Fig. 4

This was observed by Willmore [W] and lead him to the conjecture that $W \geq 2\pi^2$ for *any* torus in $\mathbb{R}^3$. The conjecture has been proved by Li and Yau [LY] for some conformal classes. L. Simon [S] showed that there exists a smooth immersion of the torus which minimizes the functional W , but the Willmore conjecture is still open. More generally, an immersion $x : M \to \mathbb{R}^3$ is called a *Willmore immersion* if it is a critical point of W , i.e. if the first variation vanishes at x . This will be computed in ch.3.

2. Oriented Moebius Geometry

The preceding paragraph showed that the geometry of oriented spheres played an important rôle for the Willmore functional. To any unoriented 2-sphere $S' \subset S^3$ we assign a point $P(S')$ in the complement of a disk in projective 4-space $\mathbb{R}P^4$, namely the center of the 3-sphere $S^\wedge$ which intersects S^3 orthogonally along S' or in other words the vertex of the cone C which is tangent to S^3 along S' (see Fig.1). If S' is a great sphere, then $S^\wedge$ becomes a 3-plane and C a cylinder, so the center is the point at ∞ in the direction of the normal vector of $S^\wedge$. If we start with the sphere

$$S_r(m) = \{x \in \mathbb{R}^3 \; ; \; \|x - m\| = r\}$$

in $\mathbb{R}^3$ and enlarge it to a 3-sphere in $\mathbb{R}^4$ with the same center and radius, then $P(\phi(S_r(m))$ is the center of the image of that 3-sphere under the inversion Φ of $\mathbb{R}^4$ which continues the stereographic projection $\phi : \mathbb{R}^3 \to S^3$ (cf. ch.1). From this we get

Fig 2

$$P(\phi(S_r(m)) = (2m, \|m\|^2 - r^2 - 1)/(\|m\|^2 - r^2 + 1)$$
$$= [m, \tfrac{1}{2}(\|m\|^2 - r^2 - 1), \tfrac{1}{2}(\|m\|^2 - r^2 + 1)]$$

where we consider $\mathbb{R}^4$ as the subset $\{[y,1] \; ; \; y \in \mathbb{R}^4\}$ of $\mathbb{RP}^4$. Introducing the Lorentzian scalar product on $\mathbb{R}^5$, namely

$$\langle X,Y\rangle = \sum_{i=1}^{4} X_i Y_i - X_5 Y_5$$

we have $S^3 = \{[X] \in \mathbb{RP}^4 \; ; \; \langle X,X\rangle = 0\}$ and $P(\phi(S_r(m)))$ is always a homogeneous vector $[Y]$ with $\langle Y,Y\rangle > 0$. Thus we may normalize the vector Y so that $\langle Y,Y\rangle = 1$. In this way, out of the line $[Y]$ we choose two 5-vectors Y and $-Y$ which now correspond to the two orientations of $S_r(m)$. We agree that negative values of r correspond to the orientation with respect to the outer normal so that $k = 1/r$ is the mean curvature with respect to the chosen normal. Putting $Y = P_+(\phi S_r(m))$ we get

$$P_+(\phi S_r(m)) = r^{-1} \; (m, \tfrac{1}{2}(\|m\|^2 - r^2 - 1), \tfrac{1}{2}(\|m\|^2 - r^2 + 1)) \; ,$$

and thus we have represented the spheres in $\mathbb{R}^3$ by points in the quadric $Q^4 = \{Y \in \mathbb{R}^5 ; \langle Y,Y\rangle = 1\}$ which is itself a Lorentzian manifold of constant curvature. If we let $r \to 0$, we have to renormalize by multiplying again with r, and thus we get

$$X = \lim_{r \to 0} r \cdot P_+(\phi S_r(x)) = (x, \tfrac{1}{2}(\|x\|^2 - 1), \tfrac{1}{2}(\|x\|^2 + 1))$$

as the representative of the point $x \in \mathbb{R}^3$. In fact, the mapping $x \to X$ is an isometry onto $L \cap \{X_5 - X_4 = 1\}$ where $L = \{X ; \langle X,X\rangle = 0\}$ is the light cone in $\mathbb{R}^5$. The following table shows how objects in $\mathbb{R}^3$ are represented in $\mathbb{R}^5$:

Sphere of radius r and center m	$r^{-1}\,(m, \tfrac{1}{2}(\|m\|^2-r^2-1), \tfrac{1}{2}(\|m\|^2-r^2+1))$
Sphere of mean curvature H through x with unit normal n	$(Hx+n, \tfrac{1}{2}(H\|x\|^2+2\langle x,n\rangle-H), \tfrac{1}{2}(H\|x\|^2+2\langle x,n\rangle+H))$
Plane through x with unit normal n	$(n, \langle x,n\rangle, \langle x,n\rangle)$
Point x	$(x, \tfrac{1}{2}(\|x\|^2-1), \tfrac{1}{2}(\|x\|^2+1))$
Incidence $x \in S$	$\langle X, P_+(\phi(S))\rangle = 0$
Moeb(3)	PO(4,1)

Here, the third line follows from the second by passing to $r \to \infty$. The fifth line follows since by definition, $P_+(\phi(S))$ belongs to the tangent space of L at X. The group PO(4,1) consists of those projective transformations of $\mathbb{RP}^4$ which preserve the sphere $S^3 = \{[X] ; \langle X,X\rangle = 0\}$. So their restrictions to S^3 are Moebius transformations since the 2-spheres (which are the intersections of S^3 with 3-planes) are mapped onto 2-spheres. Moreover, the Moebius group on S^3 is generated by all reflections at

2-spheres $S^3 \cap H$ where H is any hyperplane in $\mathbb{R}P^4$. Such inversion arises from the Lorentz reflection at the preimage of H in $\mathbb{R}^5$ which shows the last line.

A classical problem in Moebius geometry (cf. [Bo]) is the study of 2-parameter families of spheres (*sphere comgruence*). This is a smooth mapping $S : M \to \{\text{spheres in } \mathbb{R}^3\}$ where M is some 2-dimensional manifold. It is represented by a smooth mapping $Y : M \to Q^4$, $Y(m) = P(\phi(S(m)))$. A smooth map $x : M \to \mathbb{R}^3$ is called *enveloping surface* of the sphere congruence S if

(a) $$x(m) \in S(m) ,$$

(b) $$dx_m(T_mM) \subset T_{x(m)}S(m)$$

for all $m \in M$. Passing to the mapping $X : M \to L$ which corresponds to x, (a) and (b) are translated into

$$\langle X(m), Y(m)\rangle = 0 , \quad \langle dX_m(v), Y(m)\rangle = 0$$

for any $m \in M$, $v \in T_mM$. So we may characterize the enveloping surface X of a sphere congruence Y by

(E) $$\langle X,Y\rangle = 0 , \quad \langle X,dY\rangle = 0$$

where we have used that $d\langle X,Y\rangle = 0$. In other words, $X(m)$ is a null vector in $T_{Y(m)}Q^4$ which is normal to Y.

3. The conformal Gauß map

Now let $x : M \to \mathbb{R}^3$ be an immersion with unit normal field $n : M \to S^2$ and mean curvature H. The *central sphere of* x *at* $m \in M$ is the sphere $S(m)$ which is tangent to x at $x(m)$ and has mean curvature $k = H(m)$. By the remark following Lemma 1, this is the central sphere of the principal curvature spheres at m , i.e. inversion at $S(m)$ maps one principal curvature sphere onto the other. This defines a sphere congruence which is Moebius-invariantly connected with the immersion x (cf. ch.1), called the *conformal Gauß map of* x . Clearly, x is an enveloping surface of S .

As above, let $Y(m) = P(\phi(S(m))$ be the representative in Q^4 . By the table in ch.2, we have

$$Y = H\cdot X + T \tag{1}$$

where

$$X = (x,\ \tfrac{1}{2}(\langle x,x\rangle - 1),\ \tfrac{1}{2}(\langle x,x\rangle + 1)) \tag{2}$$

is the representative of x in L and

$$T = (n,\ \langle x,n\rangle,\ \langle x,n\rangle) \tag{3}$$

the representation of the tangent plane.

<u>Proposition 1.</u>

$$\langle dy,dy\rangle = (H^2 - K)\langle dx,dx\rangle$$

<u>Proof.</u> We have $\langle X,X\rangle = 0$, $\langle T,T\rangle = 1$, $\langle X,T\rangle = 0$ and therefore $\langle dX,X\rangle = \langle dT,T\rangle = 0$, $\langle dT,X\rangle = -\langle dX,T\rangle$. Moreover,

$$dX = (dx,\ \langle x,dx\rangle,\ \langle x,dx\rangle)\ , \tag{4}$$

$$dT = (dn,\ \langle x,dn\rangle,\ \langle x,dn\rangle)\ , \tag{5}$$

from which we get $\langle dX,T\rangle = \langle dx,n\rangle = 0$ and

$$\langle dX,dX\rangle = \langle dx,dx\rangle = I\ ,$$

$$\langle dX,dT\rangle = \langle dx,dn\rangle = -II\ ,$$

$$\langle dT,dT\rangle = \langle dn,dn\rangle = III\ ,$$

where I, II, III denote the classical fundamental forms. Thus

$$\langle dY, dY\rangle = H^2 \cdot I - 2H \cdot II + III = (H^2 - K) \cdot I$$

where we used the relation $III = 2H \cdot II - K \cdot I$.

Thus, Y is a conformal map with respect to the conformal structure on M induced by x, and the Willmore functional is the area of Y,

$$W(x) = \mathrm{Area}(Y) .$$

In particular, x is a Willmore immersion if Y is a minimal surface. To see also the converse, we need to compute the mean curvature vector of Y. Let Δ be the Laplace operator with respect to the metric induced by x on M.

<u>Proposition 2.</u>

$$\Delta Y + 2(H^2 - K)Y = (\Delta H + 2(H^2 - K)H)X$$

<u>Lemma :</u>

$$\begin{aligned} \Delta x &= 2H \cdot n , \\ \Delta n &= -2\, dx(\nabla H) - 2(2H^2 - K)n , \\ \Delta \rho &= 2(1 + H \cdot \sigma) , \\ \Delta \sigma &= -2 \langle \nabla H, \nabla \rho\rangle - (4H^2 - 2K)\sigma - 2H , \end{aligned}$$

where $\rho = \langle x,x\rangle/2$, $\sigma = \langle x,n\rangle$.

<u>Proof of the lemma.</u> Let e_1, e_2 be a local orthonormal basis on M. Let $\partial_i = \partial_{e_i}$ and $D_i = D_{e_i} = (\partial_i)^T$, where D is the Levi-Civita connection on M. Let us agree that we sum over repeated indices. Then

$$\Delta x = \partial_i \partial_i x - dx(D_i e_i) = \langle \partial_i dx(e_i), n\rangle n = II_{ii} \cdot n = 2H \cdot n ,$$

$$\Delta n = \partial_i\partial_i n - dn(D_i e_i) = D_i(dn)(e_i) + \langle \partial_i\partial_i n, n\rangle n$$
$$= -D_i(II)_{ij}\cdot dx(e_j) - (II_{ij})^2\cdot n$$
$$= -D_j(II)_{ii}\cdot dx(e_j) - (k_1^2 + k_2^2)n$$
$$= -2\partial_j H\cdot dx(e_j) - (4H^2 - 2K)n$$
$$= -2\ dx(\nabla H) - (4H^2 - 2K)n\ .$$

Now for any two functions $u,v : M \to \mathbb{R}^3$ we have

$$\Delta\langle u,v\rangle = \langle u,\Delta v\rangle + \langle v,\Delta u\rangle + 2\ \mathrm{trace}\ \langle du,dv\rangle\ ,$$

hence

$$\Delta\rho = \langle x,\Delta x\rangle + \mathrm{trace}\ \langle dx,dx\rangle = 2H\langle x,n\rangle + 2 = 2(1 + H\sigma)\ ,$$
$$\Delta\sigma = \langle x,\Delta n\rangle + \langle n,\Delta x\rangle + 2\ \mathrm{trace}\ \langle dx,dn\rangle$$
$$= -2\ \langle x,dx(\nabla H)\rangle - (4H^2 - 2K)\langle x,n\rangle + 2H - 4H$$
$$= -2\ d\rho(\nabla H) - (4H^2 - 2K)\sigma - 2H\ .$$

Proof of Proposition 2. We have by (1)

$$\Delta Y = \Delta(H\cdot X + T) = (\Delta H)X + H\cdot\Delta X + 2\ dX(\nabla H) + \Delta T$$

and by 3.3 and (2),(3),

$$\Delta X = \Delta(x,\rho,\rho) = 2H\cdot T + 2\cdot(0,1,1)\ ,$$
$$\Delta T = \Delta(n,\sigma,\sigma) = -(4H^2 - 2K)\cdot T - 2H\cdot(0,1,1) - 2\ dX(\nabla H),$$

hence

$$\Delta Y = (\Delta H)X - 2(H^2 - K)\cdot T$$

which gives the desired equality since $T = Y - H\cdot X$.

Now let M be a surface (which may be non-compact) and $x : M \to \mathbb{R}^3$ an immersion. We call x a *Willmore immersion* if $W(x|M')$ is stationary for any relatively compact open subset $M' \subset M$, i.e. for any smooth variation x^t of x with $x^t = x$ outside M' we have $\delta W(x|M') := (d/dt)W(x^t|M') = 0$.

Proposition 3. *The following statements are equivalent:*

(i) x *is a Willmore immersion,*

(ii) Y *is a conformal harmonic map,*

(iii) $\Delta H + 2(H^2 - K)H = 0$.

Proof. By Prop. 1 , $Y : M \to Q^4 \subset \mathbb{R}^5$ is always a conformal map. By the first variation formula (see next chapter, Equation (V)), for any variation of Y in Q^4 whose variation vector δY has compact support we have

$$\delta(\mathrm{Area}(Y)) = - \int_M \langle (\Delta Y)^{T_Q}, \delta Y\rangle \, dA .$$

where ^{T_Q} denotes the projection onto the tangent space of Q . By Prop. 2,

(a) $$(\Delta Y)^{T_Q} = h\cdot X$$

with $h = \Delta H + 2(H^2-K)H$ (recall $\langle Y,X\rangle = 0$). Moreover, $\delta Y = (\delta H)\cdot X + H\cdot\delta X + \delta T$, and $\langle X,X\rangle = 0$, $\langle X,\delta X\rangle = 0$, $-\langle X,\delta T\rangle = \langle\delta X,T\rangle = \langle\delta x,n\rangle =: f$. Hence

(b) $$\delta(\mathrm{Area}(Y)) = \int_M h\cdot f \, dA .$$

Now (i) says that $\delta(\mathrm{Area}(Y)) = 0$ for all compactly supported variation fields δx along x . Hence the integral in (b) vanishes for all f with compact support which is possible only if $h \equiv 0$. So we get (iii). But (iii) implies $(\Delta Y)^{T_Q} = 0$, by (a) , so we get (ii) which trivially implies (i).

Examples. (a) Let $x' : M \to S^3$ be a minimal immersion with unit normal vector $n' : M \to S^3$. Then the central sphere $S(p)$ for $p \in M$ is the intersection of S^3 with the hyperplane $n'(p)^\perp \subset \mathbb{R}^4$. Hence $P(S(p))$ is the point at ∞ in the direction

$n'(p)$, namely $[n'(p),0]$. Thus $Y = (n',0)$. Since n' is also a minimal immersion in S^3 (e.g. see [ET1]) , Y is a minimal surface in the totally geodesic hypersurface $S^3 = Q \cap \mathbb{R}^4$ in Q^4 . Thus x' is a Willmore immersion.

(b) By Proposition 3(iii), any minimal immersion $x : M \to \mathbb{R}^3$ is a Willmore immersion, since $H \equiv 0$. In fact, we have $Y = T$, and this has values in the totally geodesic hypersurface $Q \cap \{Y_4=Y_5\}$ with degenerate metric in Q .

4. First variation of area for conformal maps

Let M be a surface with conformal structure , $\langle\ ,\ \rangle$ a scalar product (possibly indefinite) on $\mathbb{R}^n$ and $Y : M \to \mathbb{R}^n$ a smooth mapping. For any relatively compact open subset $M' \subset M$ with conformal coordinate $(u_1,u_2) : M' \to \mathbb{R}^2$ we put

$$\mathrm{Area}(Y|M') = \int_{M'} \|Y_1 \wedge Y_2\|\, d^2u$$

where $d^2u = du_1du_2$, $Y_i = \partial Y/\partial u_i$ and

$$\|v\wedge w\| = |\langle v\wedge w, v\wedge w\rangle|^{1/2} = |\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2|^{1/2}$$

for any $v,w \in \mathbb{R}^n$. In particular, if Y is conformal, i.e. if

$$\langle Y_i, Y_j\rangle = E\cdot\delta_{ij}$$

for some smooth nonnegative function E on M' , then

$$\mathrm{Area}(Y|M') = \int_{M'} E\, d^2u\ .$$

Now let $Y^t : M \to Q$ be a smooth variation of a conformal map $Y : M \to Q$ with $Y^t = Y$ outside M' . Let $\delta Y =$

$(\partial Y^t/\partial t)|_{t=0}$. Putting $a_{ij} = \langle Y_i, \delta Y_j\rangle$ we have

$$\langle Y^t{}_i, Y^t{}_j\rangle = E\cdot\delta_{ij} + t(a_{ij} + a_{ji}) + O(t^2) ,$$

and hence

$$\|Y^t{}_1 \wedge Y^t{}_2\|^2 =$$

$$E^2(1 + O(t^2)) + 2tE(a_{11} + a_{22}) - t^2(a_{12} + a_{21})^2 + O(t^3) .$$

In those points where $E \neq 0$, we get

$$\|Y^t{}_1 \wedge Y^t{}_2\| = E(1 + 2tE^{-1}(a_{11} + a_{22}) + O(t^2))^{1/2}$$

$$= E + t(a_{11} + a_{22}) + O(t^2) ,$$

whence $\delta\|Y_1 \wedge Y_2\| = a_{11} + a_{22}$. On the other hand, where $E = 0$ we also have $a_{ij} = 0$, so $\|Y^t{}_1 \wedge Y^t{}_2\|^2 = O(t^3)$ which implies $\delta\|Y_1 \wedge Y_2\| = 0$ in these points. In particular, $A(t) := \mathrm{Area}(Y^t|M')$ is differentiable at $t=0$ with

$$\delta A = \int_{M'} (a_{11} + a_{22})\, d^2u ,$$

and since

$$a_{ij} = \langle Y_i, \delta Y_j\rangle = \langle Y_i, \delta Y\rangle_j - \langle Y_{ij}, \delta Y\rangle ,$$

we get

$$\delta A = \int_{M'} (\operatorname{div} \xi)\, d^2u - \int_{M'} \langle \Delta_u Y, \delta Y\rangle\, d^2u = - \int_{M'} \langle \Delta_u Y, \delta Y\rangle\, d^2u$$

where $\Delta_u Y = Y_{11} + Y_{22}$ and $\xi = (\langle\delta Y, Y_1\rangle, \langle\delta Y, Y_2\rangle)$. If g is any compatible metric on M with Laplacian Δ_g and area element dA_g, then $\Delta_g Y\, dA_g = \Delta_u Y\, d^2u$ on M' and hence

$$\text{(V)} \qquad \delta(\mathrm{Area}(Y)) = \int_{M'} \langle \Delta_g Y, \delta Y\rangle\, dA_g .$$

This is the first variation formula for the area of conformal maps. Using a partition of unity, we see that (V) is valid for any relatively compact open subset $M' \subset M$. If the variation Y^t takes value in some submanifold $Q \subset \mathbb{R}^n$, then δY is a tangent vector of Q and so we only need the TQ-component of $\Delta_g Y$.

5. Minimal surfaces in quadrics

Let M be a Riemann surface. Fix a (possibly indefinite) scalar product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$ and let $Y : M \to \mathbb{R}^n$ be a conformal immersion. More precisely, for any holomorphic chart $z = u_1 + iu_2$ of M we have (with a slight change of notation)

$$\langle Y_1, Y_2\rangle = 0\ , \quad \langle Y_1, Y_1\rangle = \langle Y_2, Y_2\rangle =: 2E > 0\ .$$

where $Y_i = \partial_i Y = \partial Y/\partial u_i$. Let NY be the normal bundle and $\alpha : TM \otimes TM \to NY$ the second fundamental form with the components $\alpha_{ij} = (Y_{ij})^\perp$ (where $\perp$ denotes the normal part). Then

$$\eta = (\alpha_{11} + \alpha_{22})/(4E)$$

is the mean curvature vector. Using the Wirtinger operators

$$\partial_z = \tfrac{1}{2}(\partial_1 - i\partial_2)\ , \quad \partial_{\bar z} = \tfrac{1}{2}(\partial_1 + i\partial_2)$$

we get after extending the scalar product $\langle\ ,\ \rangle$ complex bilinearly to $\mathbb{C}^n$

$$\langle Y_z, Y_z\rangle = \langle Y_{\bar z}, Y_{\bar z}\rangle = 0\ , \quad \langle Y_z, Y_{\bar z}\rangle = E\ ,$$

$$Y_{z\bar z} = \alpha_{z\bar z} = \tfrac{1}{4}\ (\alpha_{11} + \alpha_{22}) = E\cdot\eta$$

(note that $Y_{z\bar z}$ is othogonal to Y_z and $Y_{\bar z}$, hence a normal vector).

Now consider the quadric

$$Q = \{x \in \mathbb{R}^n\ ;\ \langle x,x\rangle = 1\}$$

in $\mathbb{R}^n$ and let $Y : M \to Q$ be a conformal minimal immersion. Then η is normal to Q , hence a multiple of Y , and since

$$\langle \eta, Y\rangle = \langle Y_{z\bar z}, Y\rangle/E = -\langle Y_z, Y_{\bar z}\rangle/E = -1\ ,$$

we get $\eta = -Y$, thus

$$Y_{z\bar z} = -2E\cdot Y\ .$$

On M we consider the quartic form $\langle\alpha,\alpha\rangle$. More generally, if Λ is any symmetric m-form on M , it has the coordinate representation

$$\Lambda = \sum_{j+k=m} \Lambda_{jk}\, dz^j\, d\bar{z}^k ,$$

and this decomposition is invariant under change of the holomorphic chart. We call $\Lambda^{(j,k)} = \Lambda_{jk}\, dz^j\, d\bar{z}^k$ the *(j,k)-part of* Λ .

<u>Proposition 1</u> *Let* $Y : M \to Q \subset \mathbb{R}^n$ *be a conformal minimal immersion. Then*

$$\langle\alpha,\alpha\rangle^{(4,0)} = \langle Y_{zz}, Y_{zz}\rangle dz^4 ,$$

and $\langle\alpha,\alpha\rangle^{(4,0)}$ *is a holomorphic form.*

<u>Proof.</u> We have $Y_{zz} = a\cdot Y_z + b\cdot Y_{\bar z} + \alpha_{zz}$ for some functions a, b . But since $b\cdot E = \langle Y_{zz}, Y_z\rangle = 0$, we have $b = 0$. Therefore $\langle Y_{zz}, Y_{zz}\rangle = \langle\alpha_{zz},\alpha_{zz}\rangle$. Moreover,

$$\langle Y_{zz}, Y_{zz}\rangle_{\bar z} = 2\ \langle Y_{z\bar z z}, Y_{zz}\rangle = -2\ \langle (E\cdot Y)_z, Y_{zz}\rangle = 0$$

since $\langle Y, Y_{zz}\rangle = -\langle Y_z, Y_z\rangle = 0$ and $\langle Y_z, Y_{zz}\rangle = \frac{1}{2}\langle Y_z, Y_z\rangle_z = 0$. Therefore, the form is holomorphic.

<u>Remark.</u> If Y is only a conformal harmonic map, then Y is a branched minimal immersion (cf. [GOR], [ET]). In particular, Y is an immersion outside a set of isolated points. Since the form $\langle Y_{zz}, Y_{zz}\rangle\, dz^4$ is defined everywhere, it is holomorphic also in this case.

The case $n = 5$ is of particular interest. Then NY is a 2-dimensional bundle. We may choose a local pseudo-orthonormal frame

N_1, N_2 of $NY \otimes \mathbb{C}$, i.e.

$$\langle N_1,N_1\rangle = \langle N_2,N_2\rangle = 0 \ , \quad \langle N_1,N_2\rangle = 1 \ .$$

Then we have the decomposition

$$(*) \qquad \langle Y_{zz},Y_{zz}\rangle = \langle Y_{zz},N_1\rangle\langle Y_{zz},N_2\rangle$$

and each of the factors satisfies a Cauchy-Riemann equality:

<u>Proposition 2.</u>

$$\langle Y_{zz},N_1\rangle_{\bar z} = \langle N_{1\bar z},N_2\rangle\cdot\langle Y_{zz},N_1\rangle \ ,$$

$$\langle Y_{zz},N_2\rangle_{\bar z} = -\langle N_{1\bar z},N_2\rangle\cdot\langle Y_{zz},N_2\rangle \ .$$

<u>Proof.</u> We have $\langle Y_{zz},N_1\rangle_{\bar z} = \langle (Y_{z\bar z z},N_1\rangle + \langle Y_{zz},N_{1\bar z}\rangle$. The first term vanishes since $Y_{z\bar z} = -E\cdot Y$, and N_1 is perpendicular to Y and to Y_z since N_1 is a tangent vector of Q and a normal vector of Y . Moreover, since $\langle N_{1\bar z},Y_z\rangle = -\langle N_1,Y_{z\bar z}\rangle = 0$ and $\langle N_{1\bar z},N_1\rangle = 0$, the vector $N_{1\bar z}$ lies in the span of Y_z and N_1 , due to the pseudo-orthogonality of the bases $Y_z,Y_{\bar z}$ and N_1,N_2 . But Y_{zz} is orthogonal to Y_z , hence only the N_1-component of $N_{1\bar z}$ (which is $\langle N_{1\bar z},N_2\rangle\cdot N_1$) contributes to the scalar product with Y_{zz} , whence $\langle Y_{zz},N_{1\bar z}\rangle = \langle N_{1\bar z}.N_2\rangle\cdot\langle Y_{zz},N_1\rangle$. This shows the first equation. The second follows since $\langle N_{1\bar z},N_2\rangle = -\langle N_{2\bar z},N_1\rangle$.

<u>Remark.</u> Clearly, Proposition 2 implies Proposition 1 in the 4-dimensional case. Moreover it shows that any of the functions $\langle Y_{zz},N_i\rangle$ has isolated zeros or vanishes everywhere. This remains true for the non-zero factor if the product $\langle Y_{zz},Y_{zz}\rangle$ vanishes.

<u>Proposition 3.</u> *If* $\langle Y_{zz},Y_{zz}\rangle = 0$ *, then*

$$N_{jz} \equiv 0 \ (\mathrm{mod}\ N_j)$$

either for $j=1$ *of for* $j=2$.

<u>Proof.</u> Since one of the factors $\langle Y_{zz}, N_j\rangle = -\langle Y_z, N_{jz}\rangle$ in the decomposition (*) vanishes and $\langle N_{jz}, Y_{\bar{z}}\rangle = -\langle N_j, Y_{\bar{z}z}\rangle = 0$, the tangent component of N_{jz} vanishes, and so N_{jz} is a multiple of N_j.

<u>Remark.</u> So far, we did not distinguish between different types of the scalar product $\langle\ ,\ \rangle$ in $\mathbb{R}^5$. But the consequences of the last result are different for various types. In general, Proposition 3 only says that $[N_j]$ is an antiholomorphic mapping into $\mathbb{C}P^4$, but in case of the type $(++++-)$, N_1 and N_2 are *real* vectors, and so $[N_j]$ must be a constant line, and $Y(M) \subset [N_j]^{\perp}$.

<u>Proposition 4.</u> *Let* Λ *be a holomorphic m-form on a compact Riemann surface* M *. Then either* $\Lambda \equiv 0$ *or*

$$\chi(M) = -N(\Lambda)/m$$

where $N(\Lambda)$ *is the number of zeros, counted with their order, and* $\chi(M)$ *the Euler number.*

<u>Proof.</u> Λ is a section in the complex line bundle $\otimes^m T^{+}M$ where $T^{+}M$ denotes the bundle of (1,0)-forms or the complex dual of TM, viewed as a complex line bundle. The Euler number of this bundle is $\chi(\otimes^m T^{+}M) = -m\cdot\chi(TM)$. By the index theorem of Poincarè-Hopf, this Euler number equals the index sum of the zeros of Λ, unless $\Lambda \equiv 0$. But locally, Λ behaves like a holomorphic function, and so the index of a zero is the order of that zero, which finishes the proof.

6. Willmore Spheres and Tori

Now let $x : M \to \mathbb{R}^3$ be a Willmore immersion. Then by ch.3, the conformal Gauß map $Y : M \to Q^4$ is a conformal harmonic map. If Y is not constant (which means that x is not an umbilic surface), then Y is a conformal minimal immersion on some open dense subset $M' \subset M$. In fact, the mapping $p \to dY_p(T_pM) : M' \to G_2(\mathbb{R}^5)$ can be smoothly extended to all af M (cf. [ET]) so that the normal bundle NY is defined everywhere. The induced metric on NY has type $(+-)$, and so it contains two null lines $[N_1]$, $[N_2]$ which correspond to the two enveloping surfaces, by (E), ch.2. One of these enveloping surfaces is the given immersion x, so we may assume $N_1 = X$. Bryant [Br] calls the remaining enveloping surface $X^\wedge := N_2$ the *conformal transform of* X. In the case where $\langle Y_{zz}, Y_{zz} \rangle \equiv 0$, this is a constant map, by Proposition 3 of the preceding chapter. Then $X^\wedge \in L$ represents a point $x^\wedge \in \mathbb{R}^3 \cup \{\infty\}$ which can be mapped to ∞ by a Moebius transformation $g \in Moeb(3)$ (in fact by an inversion). Since all the spheres corresponding to Y pass through $x^\wedge$, they are mapped onto planes under g. But these planes are the central spheres of the immersion $g \circ x : M \to \mathbb{R}^3 \cup \{\infty\}$, and so $g \circ x$ is a minimal immersion. Thus we have proved

<u>Proposition.</u> *Let* $x : M \to \mathbb{R}^3$ *be a Willmore immersion. Let* $Y : M \to Q^4$ *be its conformal Gauß map and suppose that* $\langle \alpha, \alpha \rangle^{(4,0)} \equiv 0$ *where* α *denotes the second fundamental form of* Y. *Then* x *is either umbilic or a Moebius transform of a minimal immersion* $x' : M \to \mathbb{R}^3 \cup \{\infty\}$.

If M is homeomorphic to a sphere, then its Euler number is postitive, and so we must have $\langle\alpha,\alpha\rangle^{(4,0)} \equiv 0$, by Prop.4 of ch.5. Hence we get a main result of [Br]:

Theorem 1 *A Willmore sphere in* $\mathbb{R}^3$ *is either umbilic or a Moebius transform of some complete minimal surface with finite total curvature.*

The last statement holds since in the minimal case, the total curvature is the Willmore functional (up to sign) which is invariant under Moebius transformations.

Moreover we note that any branch point of Y gives a zero of $\langle Y_{zz},Y_{zz}\rangle = -\langle Y_z,Y_{zzz}\rangle$. The branch points of Y in turn are umbilic points of x, by ch.3, Prop.1. Thus it follows from Prop.4 of ch.5 that a Willmore torus $x : T^2 \to \mathbb{R}^3$ cannot have umbilic points unless $\langle\alpha,\alpha\rangle^{(4,0)} = 0$. In the latter case, it follows from Osserman's theorem [O] that $W(x) \geq 8\pi > 2\pi^2$. In particular we get: wrong!

Theorem 2. *An immersed torus in* $\mathbb{R}^3$ *which minimizes the Willmore functional has no umbilic points.*

unproved!

Appendix

The functional W has been studied first by W.Blaschke [Bl] and G.Thomsen [T]. They considered it as a substitute for the area of surfaces in conformal geometry. Consequently, the surfaces x with $\delta W(x) = 0$ (Willmore immersions) were called "Konformminimalflächen" (conformally minimal surfaces). So the conformal invariance of W was clear from the beginning. Later, B.Y.Chen observes in a more general context [Cn] that W is in fact invariant not only under conformal diffeomorphisms, but under conformal changes of the metric of the target space. This is easily seen as follows: If M^n is a differentiable manifold and I some interval, consider the Riemannian metric $g = f^2dt^2 + g_t$ on $I \times M$, where f is a positive function on $I\times M$ and g_t a family of metrics on M. Then $N := f^{-1}\cdot\partial/\partial t$ is a unit normal vector field of the hypersurface $\{t\}\times M$ and if b denotes the second fundamental form of $\{t\}\times M$ with respect to $-N$, we get

$$dg_t/dt = 2f\cdot b \ .$$

Thus, if we consider a conformally changed metric $g\hat{} = \lambda^2\cdot g$ on $M\times I$, the corresponding second fundamental form $b\hat{}$ satisfies

$$2f\cdot\lambda\cdot b\hat{} = dg_t\hat{}/dt = 2\lambda\cdot(\partial\lambda/\partial t)\cdot g_t + \lambda^2\cdot 2f\cdot b \ .$$

Thus the corresponding 2^{nd} fundamental tensors B, $B\hat{}$ defined by $g_t(B(x),y) = b(x,y)$, $g_t\hat{}(B\hat{}(x),y) = b\hat{}(x,y)$ satisfy

$$\lambda\cdot B\hat{}(x) = B(x) + \mu\cdot x$$

with $\mu = f^{-1}\cdot\partial(\log\lambda)/\partial t$. Hence, if k_j are the eigenvalues of B and dv_t the volume element of g_t, then $\sum\limits_{i<j} |k_i - k_j|^n \, dv_t$ is invariant under conformal changes of the metric g.

Thomsen [T] already states the first variational formula for the functional W (cf. Prop. 3 of Ch.3) which was derived by W.Schadow. The second variation was computed much later by J.L.Weiner [We] who also observes that the Willmore torus (cf. Ch.1) is stable in the sense that the second variation of W is nonnegative. The quartic form Λ which we introduce in Ch.5 was considered by Chern [Ch] for minimal surfaces in S^n. It was the idea of R.Bryant [Br] to use this form for the conformal Gauß map of a Willmore immersion.

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(March 1988)

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