INTRODUCTION TO MARQUES' AND NEVES' SOLUTION OF THE WILLMORE CONJECTURE

J.-H. ESCHENBURG

1. Conformal geometry and the Willmore energy

Let $\Sigma \subset \mathbb{R}^3$ be a closed surface.[1] Let k_1, k_2 be the principal curvatures and $K = k_1 k_2$, $H = \frac{1}{2}(k_1 + k_2)$ Gauss and mean curvature. We will consider the expression $H^2 - K = \frac{1}{4}(k_1 - k_2)^2$. Its integral is called *Willmore energy*,

$$W_o(\Sigma) = \int_\Sigma (H^2 - K)da = \frac{1}{4}\int_\Sigma (k_1 - k_2)^2 da. \tag{1}$$

Since $\int_\Sigma K\,da = 2\pi\chi(\Sigma)$ by Gauss-Bonnet, $W(\Sigma)$ is essentially the L^2-Norm of the mean curvature function. One of its remarkable properties is its invariance under conformal deformations of the ambient space.[2] This can be seen without computation as follows. The conformal diffeomorphisms of (open subsets of) $\mathbb{R}^3$ are Moebius maps, preserving the set of spheres and planes in $\mathbb{R}^3$. These are generated by isometries, homotheties and a sphere inversion.[3] If $\Sigma \subset \mathbb{R}^3$ is a surface with normal vector N, we consider at any $p \in \Sigma$ the one-parameter family of spheres S_t through p centered on the line $t \mapsto p + tN$, including the plane through p perpendicular to N ("$t = \infty$"). When $|t|$ is small enough, S_t lies near p on one side of Σ for $t > 0$, on the other for $t < 0$. There are two positions $t_1, t_2 \in \mathbb{R} \cup \infty$ where this ceases to be true; the corresponding reciprocal values $k_i = 1/t_i$ are the principal curvatures, and the corresponding spheres are called *principal curvature spheres*. By this description it is apparent that a conformal diffeomorphism g sends principal curvature spheres of Σ onto principal curvature sphere of $g\Sigma$. Now the conformal invariance of W follows from the following Lemma.

Lemma 1.1. *Let $S_1, S_2 \subset \mathbb{R}^n$, $n \geq 3$, be spheres touching each other at some point p, that is they have a common normal vector at p. Let g be a Moebius diffeomorphism on (an open subset of) $\mathbb{R}^n$. Then*

$$\lambda|k_1' - k_2'| = |k_1 - k_2| \tag{2}$$

where k_i and k_i' are the principal curvatures of S_i and gS_i and λ is the metric dilatation factor for g at p.

Proof. The theorem is apparently true for isometries and for homotheties. It is also true for the inversion fixing p and interchanging S_1 and S_2; the dilatation factor at the fixed point p is one. Thus it is true for all Moebius diffeomorphisms. □

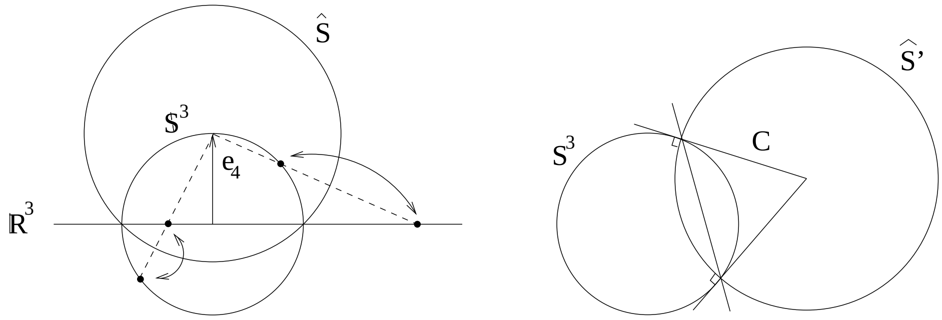
Date: December 3, 2013.

[1] See [2] for this section.

[2] W_o is invariant even under conformal changes of the ambient metric.

[3] All sphere inversions are conjugate by homotheties (changing the sphere radius) and translations (moving the center).

Using the stereographic projection $\phi : \mathbb{R}^3 \to \mathbb{S}^3$, the conformal geometry on $\mathbb{R}^3$ is transplanted to the 3-sphere $\mathbb{S}^3$. This is a conformal map which maps spheres and planes onto spheres in $\mathbb{S}^3$ and conjugates conformal maps on $\mathbb{R}^3$ and $\mathbb{S}^3$. We may think of both ϕ and ϕ^{-1} as restrictions (to $\mathbb{R}^3$, $\mathbb{S}^3 \subset \mathbb{R}^4$) of the sphere inversion at the hypersphere $\hat{S} \subset \mathbb{R}^4$ passing through $\mathbb{S}^2 = \mathbb{R}^3 \cap \mathbb{S}^3$ and centered at $e_4 = (0, 0, 0, 1)$, the only point in $\mathbb{S}^3$ which is not in the image of ϕ (left figure).



The principal curvature spheres for the image surface $\phi\Sigma \subset \mathbb{S}^3$ are defined like in $\mathbb{R}^3$, and therefore ϕ maps the principal curvature spheres S_i of $\Sigma \subset \mathbb{R}^3$ onto those of $\phi\Sigma \subset \mathbb{S}^3$. The corresponding spherical principal curvatures k_i' of $\Sigma' = \phi\Sigma$ can be seen replacing $\mathbb{S}^3$ by its tangent cone C along S_i' (which is flat in 2 dimensions); they are the euclidean principal curvatures of the "orthospheres" $\hat{S}_i' \subset \mathbb{R}^4$ through S_i' perpendicular to $\mathbb{S}^3$ and hence centered at the vertex of C (right figure). Applying Lemma 1.1 to $n = 4$ and the orthospheres $\hat{S}_i$ (perpendicular to $\mathbb{R}^3$) and $\hat{S}_i'$ (perpendicular to $\mathbb{S}^3$) which are mapped upon each other by ϕ we see that $W_o(\Sigma) = W(\Sigma')$ where

$$W(\Sigma') = \int_{\Sigma'} \frac{1}{4}(k_1' - k_2')^2 da = \int_{\Sigma'} ((H')^2 - K' + 1) da \tag{3}$$

where we have used the Gauss equation $1 = K' - k_1'k_2'$ for a surface $\Sigma' \subset \mathbb{S}^3$.

From now on we will work in the sphere $\mathbb{S}^3$ rather than in $\mathbb{R}^3$. Changing our notation, the surface Σ lies already in $\mathbb{S}^3$, and the H and K denote the spherical mean curvature and the Gauss curvature of the induced metric, respectively. Since $\int_\Sigma K\, da = 2\pi\chi(\Sigma)$ by Gauss-Bonnet, we omit K and define for $\Sigma \subset \mathbb{S}^3$ the Willmore energy

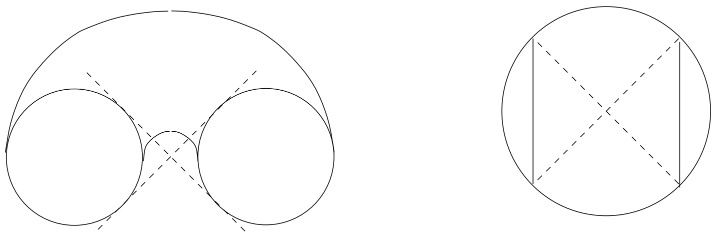
$$W(\Sigma) = \int_\Sigma (1 + H^2) da. \tag{4}$$

Clearly, $W \geq A$ where $A(\Sigma)$ denotes the area of Σ. For surfaces of genus zero, the Willmore energy is minimized by the great spheres ($H = 0$) which have $W = A = 4\pi$. The Willmore conjecture says that among surfaces of positive genus, the functional W takes its minimum precisely for the *Clifford torus* $\Sigma = \mathsf{T}$ where

$$\mathsf{T} = \mathbb{S}^1_{1/\sqrt{2}} \times \mathbb{S}^1_{1/\sqrt{2}} \subset \mathbb{S}^3 \subset \mathbb{R}^2 \times \mathbb{R}^2 .$$

Its image under stereographic projection $\mathbb{S}^3 \to \mathbb{R}^3$ is the torus of revolution whose profile circle has two tangents meeting each other perpendicularly at the origin.[4]

[4]The right figure below shows the great 2-sphere $\mathbb{S}^2 = \mathbb{S}^3 \cap \mathbb{R}^3_{134}$ orthogonally projected onto $\mathbb{R}^2_{13}$. The intersection with the Clifford torus $\mathsf{T} \cap \mathbb{S}^2 = \mathbb{S}^0_{1/\sqrt{2}} \times \mathbb{S}^1_{1/\sqrt{2}}$ are two small circles in $\mathbb{S}^2$ which are projected to the two vertical line segments. The dashed line segments denote two



To prove this conjecture we can restrict our considerations to embedded surfaces since Li and Yau [3] have shown that immersed surfaces have Willmore energy $\geq 8\pi$ which is bigger than $2\pi^2 = A(\mathsf{T}) = W(\mathsf{T})$.

2. The conformal group on $\mathbb{S}^3$

We consider the 3-sphere $\mathbb{S}^3$ as a subset of $\mathbb{R}^4 \subset \mathbb{RP}^4$. More precisely, putting $\hat{x} = (x, t) \in \mathbb{R}^4 \times \mathbb{R} = \mathbb{R}^5$, we have

$$\mathbb{S}^3 = \{[\hat{x}] : \hat{x} \in \mathbb{R}^5,\ \hat{x} \neq 0,\ \langle \hat{x}, \hat{x} \rangle_- = 0\}$$

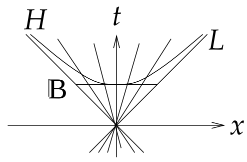
where $\langle \hat{x}, \hat{x} \rangle_- = |x|^2 - t^2$ is the Lorentzian metric on $\mathbb{R}^5$. This is the projectivized light cone $L = \{\hat{x} \in \mathbb{R}^5 : \langle \hat{x}, \hat{x} \rangle_- = 0\}$. Morover, the 4-ball $\mathbb{B}^4$ consists of the timelike homogeneous vectors,

$$\mathbb{B}^4 = \{[\hat{x}] : \langle \hat{x}, \hat{x} \rangle_- < 0\}.$$

The Lorentzian group $G = SO_{4,1} \cong O_{4,1}/\{\pm I\}$ (where $O_{4,1}$ is the group of linear transformations on $\mathbb{R}^5$ preserving the Lorentzian inner product $\langle\, , \rangle_-$) acts effectively on $\mathbb{RP}^4 = \mathbb{R}^5_*/\mathbb{R}^*$ and preserves $\mathbb{S}^3$ and $\mathbb{B}^4$. On $\mathbb{B}^4$ it acts as the isometry group with respect to the hyperbolic metric[5] and on $\mathbb{S}^3 = \partial\mathbb{B}^4$ as the conformal group (Moebius group).[6]

perpendicular great circles through $e_4 = (0, 0, 0, 1)$ (the black point in the center) tangent to $\mathsf{T} \cap \mathbb{S}^2$. The stereographic projection $\phi : \mathbb{S}^3 \setminus \{e_4\} \to \mathbb{R}^3$ from e_4 commutes with the rotations in the 12-plane and maps $\mathbb{S}^2$ onto the 13-plane in the left figure. Since T is invariant under rotations in the 12-plane, its image under ϕ is a torus of revolution. The dashed great circles through e_4 are mapped onto lines through $\phi(-e_4) = 0$ in the 13-plane; these are the dashed lines in the left figure.

[5]We may identify $\mathbb{B}^4$ with the upper sheet of the 2-sheeted hyperboloid $H = \{\hat{x} : x^2 - t^2 = -1\}$ by choosing for each homogeneous vector $[\hat{x}] \in \mathbb{B}^4$ the representative $\hat{x}$ with $\langle \hat{x}, \hat{x} \rangle_- = -1$ and $t > 0$. The hyperbolic metric on $H \subset \mathbb{R}^5$ is induced by the Lorentzian inner product on $\mathbb{R}^5$. It is clearly invariant under G. This is the Klein model of hyperbolic geometry where geodesics become straight line segments in $\mathbb{B}$.



[6]The Lorentian inner product induces on the projectivized light cone $\mathbb{S}^3 = [L]$ not a metric but only a conformal class of metrics. Each intersection of L with a spacelike hyperplane is a system of representatives for $\mathbb{S}^3$ which inherits a positive definite metric h from the ambient Lorentzian inner product. However, this metric depends on the height of the intersection for each generating line of L, but it does not depend on the slope of the intersecting hyperplane since adding a vector parallel to the generating line (being the kernel of the metric) does not change the inner product. Thus the metric h itself may be not invariant under G, but its conformal class is.

3. Willmore energy and area

Let $\Sigma \subset \mathbb{S}^3$ be a compact surface, da its surface element. Recall the Willmore energy and the area of Σ:

$$W(\Sigma) = \int_\Sigma (1 + H^2)da, \quad A(\Sigma) = \int_\Sigma da.$$

Let Σ_t be the parallel surface of (signed) distance t from Σ.

Proposition 3.1. [6]

$$W(\Sigma) \geq A(\Sigma_t) \tag{5}$$

Proof. Let $x : \Sigma \to \mathbb{S}^3$ be the embedding (or immersion) and N its normal vector field on $\mathbb{S}^3$. Then $x_t = cx + sN$ mit $c = \cos t$ and $s = \sin t$ is the parallel surface. If we fix $p \in \Sigma$ and let $v_i \in T_p\Sigma$ $(i = 1, 2)$ be the eigenbasis of the Weingarten map $\nabla N = (\partial N)^{T\mathbb{S}^3}$ with eigenvalues (principal curvatures) $-k_i$, then ∂x_t has eigenvalues $c - sk_i$ with parallel eigenvectors $v_i(t)$ along the great circle $t \mapsto x_t(p)$. Thus $A(\Sigma_t) = \int_\Sigma a(t)da$ where $a(t)$ is the Jacobian of x_t, the product of the eigenvalues of ∂x_t, and we have

$$\begin{aligned}
a(t) &= (c - sk_1)(c - sk_2) \\
&= c^2 - 2csH + s^2 k_1 k_2 \\
&\leq c^2 - 2csH + s^2 H^2 \\
&= (c - sH)^2 \\
&= \langle \left(\begin{smallmatrix} c \\ s \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ -H \end{smallmatrix}\right) \rangle^2 \\
&\leq |\left(\begin{smallmatrix} c \\ s \end{smallmatrix}\right)|^2 \left|\left(\begin{smallmatrix} 1 \\ -H \end{smallmatrix}\right)\right|^2 \\
&= 1 + H^2
\end{aligned}$$

where $H = \frac{1}{2}(k_1 + k_2)$ is the mean curvature. □

Remark. Σ_t is smooth only for small $|t|$, up to the first focal point of Σ, but at a focal point and beyond, $\Sigma_t = \partial B_t(\Sigma)$ looses even more area.

4. The canonical family

One decisive idea of the proof is a mountain pass argument. If we want to cross a mountain chain, we are looking for the lowest pass across the mountains. This means, among all possible paths across the mountains we are looking for the one whose maximal height is as small as possible. In mathematic terms, we are using a homotopy of paths and minimize the maximal height among all paths in the homotopy in order to detect a critical point with index one in our landscape ("minimax method").

We apply this idea to our situation as follows: Our "landscape" is the set of closed surfaces in $\mathbb{S}^3$ and the "height" is the area of such a surface. The minimax method is appropriate since minimal surfaces in $\mathbb{S}^3$ are never minima of the area functional: they have always positive index which means that they can be made smaller by deformation in certain directions. In fact, according to a theorem of Urbano [7], the index of a closed minimal surface Σ of positive genus in $\mathbb{S}^3$ is at least 5 with equality if and only if Σ is the Clifford torus T. At a mountains pass there is just one direction downwards, but here we have 5 such directions. Therefore our "path" must be 5-dimensional rather than 1-dimensional. Now we will describe such a 5-parameter family of surfaces, called *canonical family* which we assign to each surface $\Sigma \subset \mathbb{S}^3$.

Let G be the group of conformal transformations on $\mathbb{S}^3$ and K the subgroup of isometries of $\mathbb{S}^3$; the quotient G/K is the hyperbolic 4-ball $\mathbb{B}^4$. Since we are not interested in isometric motions, we work modulo K and obtain a 4-parameter family $g\Sigma$, $g \in \mathbb{B}^4 \subset G$ (Cartan embedding, see section 7). All $g\Sigma$ have the same Willmore energy (which is constant under conformal changes of the ambient space). Additionally we have the family of parallel hypersurfaces Σ_t, $t \in I = (-\pi, \pi)$ where we know that $W(\Sigma) \geq A(\Sigma_t)$ (Prop. 3.1). Thus

$$W(\Sigma) \geq A(\Sigma') \tag{6}$$

for all Σ' in the canonical family. We will reparametrize this family in a subtle way (see section 8) such that it can be extended to the boundary of $\mathbb{B}^4 \times I$. Hence we obtain a map

$$\Phi : I^5 \to \{\text{surfaces in } \mathbb{S}^3\}$$

(*canonical family of* Σ) which assigns to each $x \in \bar{\mathbb{B}}^4 \times \bar{I}$ a surface $\Phi(x)$, and with Prop. 3.1 we obtain

$$W(\Sigma) \geq A(\Phi(x)) \tag{7}$$

for all $x \in I^5$. We call two such maps Φ_o, Φ_1 *homotopic* if there is a homotopy between them fixing the boundary ∂I^5. Each homotopy class Π has a *width* $L(\Pi)$ which is

$$L(\Pi) = \inf_{\Phi \in \Pi} \sup_{x \in I^5} A(\Phi(x)) \tag{8}$$

We will show that $L(\Pi) > 4\pi$ when Σ has positive genus ("positive genus theorem", section 10). Further we have to assume from Geometric Measure Theory [1, 5] that the inf-sup in (8) is attained ("Min-max Theorem"): There exists a minimal surface Σ_Π with

$$A(\Sigma_\Pi) = L(\Pi). \tag{9}$$

However it is important for the proof that the min-max theory of Almgren and Pitts is more general: The previous knowledge of the index of the critical point is not needed. Hence we may replace 5 by any n. All what is needed in order to find a minimal surface Σ_Π with (9) is that the area is strictly smaller at the boundary of I^n,

$$L(\Pi) > \sup_{x \in \partial I^n} A(\Phi(x)) \tag{10}$$

for some $\Phi \in \Pi$.

5. The Clifford torus has least area

Let Σ be the closed minimal surface in $\mathbb{S}^3$ with least area among all minimal surfaces with positive genus (exists by Geometric Measure Theory). We will use a theorem of Urbano [7] stating that a closed minimal surface of positive genus has index 5 (the least possible index) if it is a Clifford torus. Thus we argue by contradiction and assume that the index of Σ is at least 6. According to (7), the canonical family Φ of Σ satisfies

$$\sup_x A(\Phi(x)) \leq W(\Sigma) = A(\Sigma)$$

(since Σ is minimal, $H = 0$, we have $W = A$), hence the supremum is a maximum,

$$\sup_x A(\Phi(x)) = A(\Sigma).$$

Since the index of $\Sigma = \Phi(x_o)$ is bigger than 5, we can deform is slightly to another surface Σ' (away from the Φ) with smaller area; in fact we can deform the whole family Φ in a small neighborhood of x_o such that the deformed family Φ' (which is no longer "canonical")[7] satisfies

$$\sup_x A(\Phi'(x)) < A(\Sigma).$$

and $\Phi' = \Phi$ on ∂I^5. (Φ is like a path leading over the top of a mountain: then nearby there is another path (circumventing the top) whose maximal height is lower.) Let $\Pi' = [\Phi']$ and $\Sigma' = \Sigma_{\Pi'}$ its minimax surface. Then

$$A(\Sigma') = L(\Pi') \leq \sup_x A(\Phi'(x)) < A(\Sigma).$$

Since $A(\Sigma') = L(\Pi') > 4\pi$ (positive genus theorem), Σ' is a minimal surface of positive genus with $A(\Sigma') < A(\Sigma)$, but Σ has already least area among those surfaces, a contradiction!

6. The Clifford torus minimizes Willmore

We start with a surface Σ with positive genus and $W(\Sigma) < 8\pi$ (which is no restriction since $W(\mathsf{T}) = 2\pi^2 < 8\pi$). We form its canonical family Φ and the homotopy class $\Pi = [\Phi]$. Then

$$L(\Pi) \leq \sup_{x \in I^5} A(\Phi(x)) \leq W(\Sigma) < 8\pi \tag{11}$$

Let Σ_Π be the minimax surface for Π, i.e. $A(\Sigma_\Pi) = L(\Pi) > 4\pi$. Then $A(\Sigma_\Pi)$ lies strictly between 4π and 8π, therefore Σ_Π cannot be a sphere (minimal spheres are great spheres, according to a theorem of H. Hopf, and therefore they have area 4π), nor it has higher multiplicity (then the area would be $\geq 8\pi$). But among the remaining surfaces, the Clifford torus has least area, hence

$$2\pi^2 \leq A(\Sigma_\Pi) = L(\Pi) \overset{(11)}{\leq} W(\Sigma) \tag{12}$$

which shows $W(\Sigma) \geq W(\mathsf{T})$.

If for some Σ with positive genus we have equality, $W(\Sigma) = W(\mathsf{T}) = 2\pi^2$, the canonical family $\Phi(x) = \Sigma_x$ satisfies $A(\Sigma_x) \leq W(\Sigma_x) \leq W(\Sigma) = 2\pi^2$. If $\sup_x A(\Sigma_x) < 2\pi^2$, we have $L(\Pi) < 2\pi^2$ for $\Pi = [\Phi]$, but then the minimax surface Σ' of Φ is a minimal surface with $A(\Sigma') < A(\mathsf{T})$ which is impossible. Thus $\sup_x A(\Sigma_x) = 2\pi^2$, and the supremum is attained at a surface $\Sigma' = \Sigma_{x_o}$ with $A(\Sigma') = 2\pi^2 = W(\Sigma')$; note that Σ' is a conformal image of Σ and not a parallel surface since from the proof of Prop. 3.1 we see that a parallel surface Σ'' would satisfy $A(\Sigma'') < W(\Sigma) = 2\pi^2$. But $A(\Sigma') = W(\Sigma')$ implies that Σ' is a minimal surface of positive genus, hence $\Sigma' = \mathsf{T}$ by section 5. Thus Σ is conformally equivalent to T.

---

[7] However, the "Positive Genus Theorem" still holds for this family Φ' since its boundary is still that of Φ which is canonical, see section 10.

7. Boundary of the canonical family

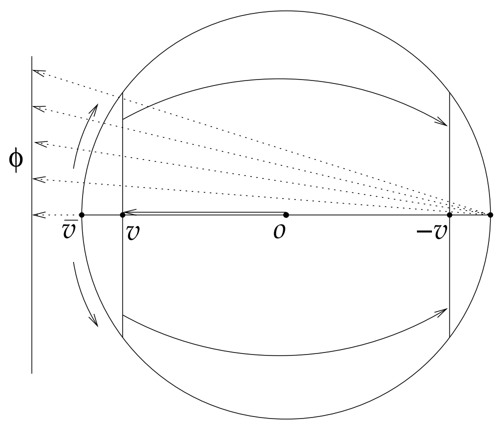
What happens with $g\Sigma$ when $g \in \mathbb{B}^4 \subset G$ approaches the boundary $\mathbb{S}^3 = \partial\mathbb{B}^4$? The embedding $\mathbb{B}^4 \hookrightarrow G$ is the Cartan embedding:

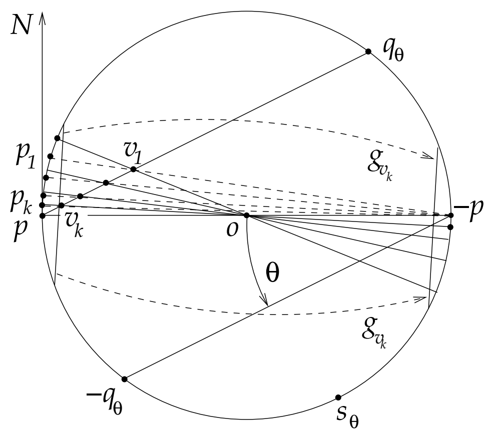
$$v \ni B^4 \mapsto g_v = s_o s_v = s_{-v} s_o \in G \tag{13}$$

where s_o and s_v are the geodesic reflections (symmetries) at the points $o, v \in \mathbb{B}^4$ with the hyperbolic metric on $\mathbb{B}^4$ where we choose o to be the origin of $\mathbb{B}^4$. The hyperbolic isometry g_v translates the geodesic $\gamma_v = vo$ through v and o (the "axis") and parallel-translates the hyperplanes intersecting γ_v perpendicularly. In particular,

$$g_v(v) = s_o s_v(v) = s_o(v) = -v. \tag{14}$$

On the sphere $\mathbb{S}^3 = \partial\mathbb{B}^4$, the hyperbolic isometry g_v acts as a conformal map: a spherical dilatation fixing the points $\pm\bar{v}$ with $\bar{v} = v/|v|$ and moving every point $q \in \mathbb{S}^3$ away from $\bar{v}$ towards $-\bar{v}$ along the great circle from $\bar{v}$ to $-\bar{v}$ through q. If $|v| \nearrow 1$, then g_v maps $\mathbb{S}^3 \setminus B_\epsilon(\bar{v})$ onto $B_\epsilon(-\bar{v})$. Consequently, if a surface $\Sigma \subset \mathbb{S}^3$ lies completely outside $B_\epsilon(\bar{v})$, it is mapped into $B_\epsilon(-\bar{v})$ and its limit as $|v| \nearrow 1$ will be only the point $-\bar{v}$. On the other hand, if $\bar{v} \in \Sigma$, then g_v acts as a dilatation on Σ. In fact, after conjugation with the stereographic projection ϕ from $-\bar{v}$ it becomes a euclidean dilatation (homothety), and a very small portion $\Sigma \cap B_\epsilon(\bar{v})$ will be enlarged by an arbitrarily large factor as $|v| \to 1$. Therefore $\phi(g_v\Sigma \cap B_\epsilon(\bar{v}))$ is converging to a plane through the origin, the tangent plane of $\Phi(\Sigma)$ at $\phi(\bar{v}) = 0$, and in the limit $|v| \to 1$, $\Sigma \cap B_\epsilon(\bar{v})$ is mapped onto the great sphere in $\mathbb{S}^3$ which is tangent to Σ at $\bar{v}$ (see left figure).





We yet have to determine $\lim g_{v_k}\Sigma$ for a sequence $(v_k)_{k\in\mathbb{N}}$ in $\mathbb{B}^4$ with $v_k \to p \in \Sigma$ (see right figure). We will restrict our attention to the normal plane P spanned by the vector p and the normal vector N of Σ at p. We show that $g_{v_k}\Sigma$ may converge to any (small or great) sphere in $\mathbb{S}^3$ which intersects P perpendicularly at $-p$.

Lemma 7.1. *Let $q_\theta \in \mathbb{S}^3 \cap P$ be the point with $\angle(-p, p, q) = \theta$ for arbitrary $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Let (v_k) be a sequence on the line seqment $[p, q_\theta]$ with $v_k \to p$, and let $p_k \in \mathbb{S}^3$ be the end point of the geodesic (line segment in $\bar{\mathbb{B}}^4$) from $-p$ through v_k. Let $S_\theta = \mathbb{S}^3 \cap H_\theta$ (with center s_θ) where $H_\theta \subset \mathbb{R}^4$ is the hyperplane intersecting P perpendicularly along the line segment $[-p, -q_\theta]$. Then*

$$\lim_{k\to\infty} g_{v_k}\Sigma = S_\theta. \tag{15}$$

Proof. Since $v_k \in P$, the isometries g_{v_k} preserve P, and

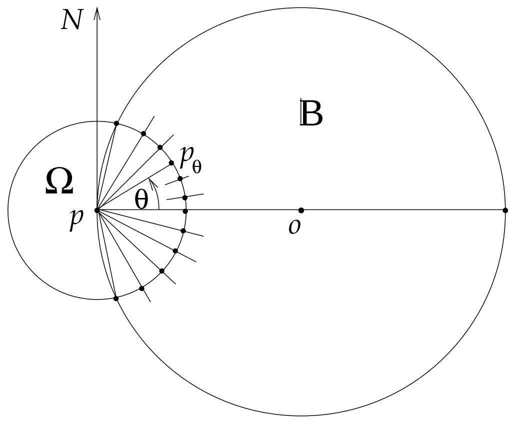
$$\begin{aligned} g_{v_k}(p) &= s_o s_{v_k}(p) \ = s_o(q_\theta) = -q_\theta, \\ g_{v_k}(-p) &= s_o s_{v_k}(-p) = s_o(p_k) = -p_k \to -p\,. \end{aligned}$$

Hence $\lim_k g_{v_k}[p,-p] = [-q_\theta,-p]$. Thus, in the limit $k \to \infty$, the hyperplane $H = N^\perp$ through $[p,-p]$ is mapped onto the hyperplane H_θ through $-p$ and $-q_\theta$ perpendicular to P, and the same holds for the intersections of these hyperplanes with $\mathbb{S}^3$. Thus in the limit $k \to \infty$, the great sphere $H \cap \mathbb{S}^3$ (in fact, an arbitrary small neighborhood of p in this great sphere) is mapped onto the small sphere $S_\theta = H_\theta \cap \mathbb{S}^3$ throught $-p$ and $-q_\theta$. Since Σ is approximated by $H \cap \mathbb{S}^3$ in a small neighborhood of p, we have shown that $\lim g_k \Sigma = S_\theta$. □

8. Continuous extension to the boundary

We would like to extend the 4-parameter family of surfaces $\{g_v\Sigma : v \in \mathbb{B}^4\}$ to the boundary of $\mathbb{B}^4$. However, as Lemma 7.1 shows, the limit is not unique as v approaches a boundary point $p \in \mathbb{S}^3 = \partial\mathbb{B}^4$ whenever $p \in \Sigma$. Therefore we have to blow up these boundary points: each $p \in \Sigma$ will be replaced by a one-parameter family of points p_θ, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. More precisely, we choose an open tubular neighborhood Ω of $\Sigma \subset \mathbb{R}^4$ with small radius ϵ such that $\partial\Omega$ is smooth. We construct a continuous map $T : \bar{\mathbb{B}}^4 \to \bar{\mathbb{B}}^4$ as follows:[8]

(1) T maps $\mathbb{B}^4 \setminus \Omega$ homeomorphically onto $\mathbb{B}^4$,
(2) T maps $\bar\Omega \cap \bar{\mathbb{B}}^4$ onto Σ by the nearest point projection π,
(3) The map $v \mapsto C(v) := g_{T(v)}\Sigma$, $v \in \mathbb{B}^4 \setminus \bar\Omega$, admits a continuous extension to $\bar{\mathbb{B}}^4$ which is constant along the projection lines of π.



The blow-up of $p \in \Sigma$ is the one-parameter family $p_\theta \in \partial\Omega \cap \bar{\mathbb{B}}^4 \cap P$ (where $P = \mathrm{Span}\,(p,N)$), parametrized by the angle $\theta \in [-\frac{\pi}{2}+\epsilon, \frac{\pi}{2}-\epsilon]$.[9] Hence we may put $C(p_\theta) = S_\theta$ which means by continuity that $T|_{\mathbb{B}^4\setminus\bar\Omega}$ eventually maps the radial lines emenating from p approximately onto the line segments $[p, q_\theta]$. For any other $q \in \mathbb{S}^3 \setminus \Sigma$, the "surface" $C(q)$ will be the trivial, $C(q) = \{-q\}$. Thus our map C is

[8] Using the line segments $\gamma_\theta = [p, q_\theta]$, parametrized on $[0, d_\theta]$ by euclidean arc length with $d_\theta = |p - q_\theta|$, we may put $T(\gamma_\theta(t)) = p$ for $t \le \epsilon$ and $T(\gamma_\theta(t)) = \gamma_\theta(\varphi(t))$ for $t \ge \epsilon$, here φ is continuous with $\varphi(t) = t$ for $t \ge 2\epsilon$ and where φ maps the interval $[\epsilon, d_\theta]$ strictly monotoneously onto $[0, d_\theta]$.

[9] The angle interval is slightly smaller than $[-\frac{\pi}{2}, \frac{\pi}{2}]$ because of the corners.

defined on $\bar{\mathbb{B}}^4$. Last we let $C(v)_t \subset \mathbb{S}^3$ be the surface of signed spherical distance t from $C(v)$, for any $v \in \bar{\mathbb{B}}^4$. Thus we have obtained a 5-parameter family of surfaces $C(v)_t$ for $(v,t) \in \bar{\mathbb{B}}^4 \times [-\pi,\pi] \cong I^5$. Note that all surfaces at the boundary are round spheres: $C(v)$ and $C(v)_t$ are round spheres for all $v \in \partial\mathbb{B}^4$ and $t \in [-\pi,\pi]$, and when $|t| = \pi$, then $C(v)_t$ is a single point or empty for any $v \in \bar{\mathbb{B}}^4$. In particular, any equidistant family $t \mapsto C(v)_t$ with $v \in \mathbb{S}^3 = \partial\mathbb{B}^4$ is a family of concentric round spheres. There is some $t \in [-\pi,\pi]$ such that $C(v)_s = \emptyset$ for all $s < t$ while $C(v)_t$ is a single point $\bar{Q}(v)$, the common center of the family. This defines a continuous map

$$\bar{Q} : \mathbb{S}^3 \to \mathbb{S}^3 \tag{16}$$

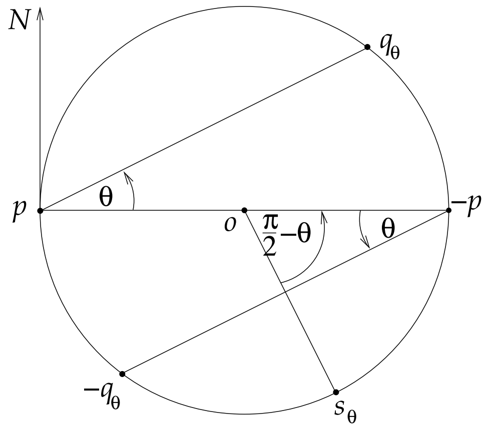
called *center map* or *extended Gauss map.*[10] The concentric sphere family $C(v)_t$ contains precisely one great sphere $C(v)_{t(v)}$. After a reparametrization (called $\tilde{\Phi}$), the t-intervall is $I = [0,1]$ and $t(v)$ is always its mid point $\frac{1}{2}$. Since the area of a sphere in $\mathbb{S}^3$ is $\leq 4\pi$ = area of a great sphere, we see

$$\sup_{x \in \partial I^5} A(\Phi(x)) = 4\pi \tag{17}$$

where $\Phi : I^5 \to \{\text{surfaces in } \mathbb{S}^3\}$ is the reparametrization of C on I^5.

9. The extended Gauss map

Let $\Sigma \subset \mathbb{S}^3$ be a closed surface with normal field N. We want to compute the extended Gauss map $\bar{Q}$ of the canonical family of Σ. The oriented closed surface Σ decomposes $\mathbb{S}^3 \setminus \Sigma$ into two connected components A and A^* where the chosen normal vector N points into A^*. Further, there is the tubular neighborhood $\Omega^S = \Omega \cap \mathbb{S}^3$ of Σ which meets both A and A^*. The map $T : \mathbb{S}^3 \to \mathbb{S}^3$ maps the closure $\bar{\Omega}^S$ radially onto Σ while the domains $\tilde{A} = A \setminus \bar{\Omega}^S$ and $\tilde{A}^* = A^* \setminus \bar{\Omega}^S$ are mapped diffeomorphically onto A and A^*. The extended Gauss map $\bar{Q} : \mathbb{S}^3 \to \mathbb{S}^3$ is as follows:



$$\bar{Q}(v) = \begin{cases} T(v) & \text{if} \quad v \in \tilde{A}, \\ -T(v) & \text{if} \quad v \in \tilde{A}^* \\ s_\theta & \text{if} \quad v = q_\theta \in \Omega^S \end{cases} \tag{18}$$

[10] $\bar{Q}$ restricted to Σ is the classical Gauss map $-N$.

Putting $c = \cos\theta$ and $s = \sin\theta$ we see from the figure above

$$s_\theta = -cN_p - sp. \tag{19}$$

Lemma 9.1. *Let $\Sigma \subset \mathbb{S}^3$ be a closed oriented surface and $\bar{Q} : \mathbb{S}^3 \to \mathbb{S}^3$ its extended Gauss map. Then the mapping degree of $\bar{Q}$ is the genus of Σ.*

Proof. Let ω be the volume form of $\mathbb{S}^3$. By definition, the mapping degree of $\bar{Q}$ is $\deg(\bar{Q}) = \int_{\mathbb{S}^3} \bar{Q}^*\omega \,/\, \int_{\mathbb{S}^3} \omega$ where the denominator is the volume of $\mathbb{S}^3$ (which is $2\pi^2$). In the numerator, the domain $\mathbb{S}^3$ will be decomposed into the three subdomains according to (18). Clearly, since T maps $\tilde{A}$ diffeomorphically onto A and $\tilde{A}^*$ onto A^*,

$$\begin{aligned}\int_{\tilde{A}} \bar{Q}^*\omega &= \int_{\tilde{A}} T^*\omega = \mathrm{vol}(A),\\ \int_{\tilde{A}^*} \bar{Q}^*\omega &= \int_{\tilde{A}^*} T^*\omega = \mathrm{vol}(A^*).\end{aligned}$$

Thus these two terms add up to vol $\mathbb{S}^3 = 2\pi^2$.

In order to compute the remaining term $\int_{\Omega^S} \bar{Q}^*\omega$, we choose an oriented orthonormal eigenbasis e_1, e_2 with $D_{e_i}N = -k_i e_i$ where D is the covariant derivative on $\mathbb{S}^3$ and k_i are the principal curvatures of $\Sigma \subset \mathbb{S}^3$. Further let $e_\theta = \partial/\partial\theta$. From

$$\bar{Q} = -cN - sp$$

where p is the position vector and N the normal vector of Σ, we obain for $i = 1, 2$

$$\begin{aligned} D_{e_i}\bar{Q} &= (k_i c - s)e_i\\ D_{e_\theta}\bar{Q} &= sN - cp\end{aligned}$$

Note that $sN - cp \in T_{\bar{Q}(p)}\mathbb{S}^3$, but the basis $(e_1, e_2, sN - cp, -cN - sp)$ has negative orientation since $\left|\begin{smallmatrix} s & -c\\ -c & -s\end{smallmatrix}\right| = -1$, thus $\omega_{\bar{Q}}(e_1, e_2, sN - cp) = -1$. Thus

$$\begin{aligned} -\omega_{\bar{Q}}(D_{e_1}\bar{Q},\, D_{e_2}\bar{Q},\, D_{e_\theta}\bar{Q}) &= (k_1c - s)(k_2c - s)\\ &= k_1k_2c^2 + s^2 - (k_1 + k_2)sc\\ &= (K-1)c^2 + s^2 - 2Hsc\\ &= Kc^2 - c_2 - Hs_2,\end{aligned}$$

with $c_2 = \cos 2\theta$ and $s_2 = \sin 2\theta$. Thus

$$\int_{\Omega^S} \bar{Q}^*\omega = -\int_\Sigma \int_{-\pi/2}^{\pi/2} (Kc^2 - c_2 - Hs_2)\, d\theta\, du$$

The functions c_2, s_2 depend only on θ while K, H depend only on $u \in \Sigma$. But note that $\int_{-\pi/2}^{\pi/2} c_2 = 0 = \int_{-\pi/2}^{\pi/2} s_2$ (integration over the full period) while $\int_{-\pi/2}^{\pi/2} c^2 = \pi/2$. On the other hand, $\int_\Sigma K = 2\pi(2 - 2g)$ where g denotes the genus of Σ. Hence

$$\int_{\Omega^S} \bar{Q}^*\omega = -2\pi^2(1 - g).$$

Now the full integral is

$$\int_{\mathbb{S}^3} \bar{Q}^*\omega = 2\pi^2 - 2\pi^2(1-g) = 2\pi^2 g.$$

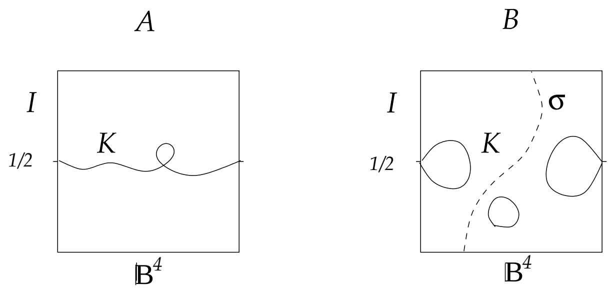
Thus we have shown $\deg \bar{Q} = g$. $\square$

10. The canonical family has width $> 4\pi$ (Positive Genus Theorem, rough idea)

Suppose not: $L([\Phi]) = 4\pi$. For the sake of exposition suppose that

$$\sup_{x \in I^5} A(\Psi(x)) = 4\pi$$

for some $\Psi \in [\Phi]$ (really this value can only be approximated). Let $\mathcal{T}$ denote the set of oriented great spheres in $\mathbb{S}^3$; we have $\mathcal{T} \cong \mathbb{S}^3$. Let $K = \Psi^{-1}(\mathcal{T})$. By our normalization of the canonical family, $\partial K \subset \mathbb{S}^3 \times \{\frac{1}{2}\}$. Now two alternatives can happen for K:



In Case A, $\mathbb{S}^3 \times \{\frac{1}{2}\}$ is the boundary of K (which consists of great spheres). Thus the map $\mathbb{S}^3 \to \mathbb{S}^3$, $v \mapsto \Psi(v, \frac{1}{2})$ has trivial homology. But this is the extended Gauss map which has degree g which is nontrivial for $g \geq 1$, a contradiction.

In Case B, there is a curve σ in $\mathbb{B}^4 \times I$ from $\mathbb{B}^4 \times \{0\}$ to $\mathbb{B}^4 \times \{1\}$ which avoids K (dotted line). The surfaces corresponding to the points of σ satisfy

$$4\pi \leq \sup A(\Psi \circ \sigma) \leq \sup A(\Psi) \leq 4\pi$$

but at the boundary of the parameter intervall, $\Psi \circ \sigma$ is the zero surface. Thus we may apply the min-max theorem on $\Psi \circ \sigma$ (see end of section 4) to conclude that there is a minimal surface with area 4π among these surfaces. But this must be a great sphere, hence σ intersects K, a contradiction!

References

[1] F. Almgren, The theory of varifolds. Mimeographed notes, Princeton (1965)
[2] J.-H. Eschenburg: Willmore surfaces and Moebius geometriy (1987), www.math.uni-augsburg.de/∼eschenbu/
[3] P. Li, S.T. Yau: A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces, Invent. Math. 69 (1982), 269 - 291
[4] F.C. Marques, A. Neves: Min-max theory and the Willmore conjecture, arXiv: 1202.6036, to appear in Ann. of Math.
[5] J. Pitts: Existence and regularity of minimal surfaces on Riemannian manifolds, Mathematical Notes 27, Princeton University Press, Princeton, (1981).
[6] A. Ros: The Willmore conjecture in the real projective space, Math. Res. Lett. 6 (1999), 487 - 493
[7] F. Urbano: Minimal surfaces with low index in the three-dimensional sphere, Proc. Am. Math. Soc. 108 (1990), 989 - 992

E-mail address, (Eschenburg): `eschenburg@math.uni-augsburg.de`

Institut für Mathematik, Universität Augsburg, D-86135 Augsburg, Germany