PSEUDOHOLOMORPHIC CURVES IN $S^6$ AND $S^5$

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Abstract. The octonian cross product on $\mathbb{R}^7$ induces a nearly Kähler structure on $S^6$, the analogue of the Kähler structure of $S^2$ given by the usual (quaternionic) cross product on $\mathbb{R}^3$. Pseudoholomorphic curves with respect to this structure are the analogue of meromorphic functions. They are (super-)conformal minimal immersions. Using a slightly different method we reprove a theorem of Hashimoto [9] giving an intrinsic characterization of pseudoholomorphic curves in $S^6$ and (beyond Hashimoto’s work) $S^5$. Instead of the Maurer-Cartan equations we use an embedding theorem into homogeneous spaces (here: $S^6 = G_2/SU_3$) involving the canonical connection. The integrability conditions must be checked only for a $3 \times 3$ matrix system instead of $7 \times 7$.

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1. Introduction

Minimal surfaces in the round 3-sphere $S^3$ have an intrinsic characterization: A metric $ds^2$ on a simply connected Riemann surface $M$ is the induced metric of a full conformal minimal immersion into $S^3$ if and only if its Gaussian curvature $K$ satisfies $K \leq 1$ and

$$\Delta \log(1 - K) = 4K$$

where $\Delta$ is the Laplacian of the metric $ds^2$. The formula goes back to Ricci [11, p. 340] who actually looked at surfaces of constant mean curvature 1 in euclidean 3-space, but these are isometric to minimal surfaces in $S^3$. There are similar (“Ricci-like”) formulas in other situations. In $S^4$, superminimal surfaces (those with trivial associated family) are characterized by the equation (cf. [7, p. 191])

$$\Delta \log(1 - K) = 6K - 2.$$  

In the present paper, we give such characterizations for certain types of minimal surfaces in $S^5$ and $S^6$:

\begin{equation}
(53) \quad \Delta \log(1 - K) = 6K
\end{equation}

for so called pseudoholomorphic curves\(^2\) in $S^5$ and

\begin{equation}
(50) \quad \Delta \log(1 - K) = 6K - 1
\end{equation}

for superminimal pseudoholomorphic curves in $S^6$ (see below). General pseudoholomorphic curves in $S^6$ allow a similar characterization [9] which however depends on an additional structure, a holomorphic 6-form $\Lambda$ on $M$ (which is zero in the superminimal case):

\begin{equation}
(46) \quad \Delta \log(1 - K) - (6K - 1) = |\Lambda|^2/(1 - K)^2.
\end{equation}

A general theory of minimal surfaces in spheres allowing for Ricci-like characterizations was recently given in [14].

Pseudoholomorphic curves in $S^6$ are the analogues of meromorphic functions on Riemann surfaces when $\mathbb{H}$ is replaced by $\mathbb{O}$. In fact, let $S \in \{S^2, S^6\}$ be the unit sphere in the imaginary quaternions $\mathbb{H}'$ or octonions $\mathbb{O}'$, respectively. Left translation with the position vector $p \in S$ induces an almost complex structure on $S$ (which is integrable for $S = S^2$). For any Riemann surface $M$, a smooth mapping $f : M \to S$ is pseudoholomorphic if its derivative $df_u : T_u M \to T_{f(u)}S^6$ is complex linear with respect to this almost complex structure. For

\(^1\)This condition makes sense even at the zeros of $1 - K$. In fact, for a minimal surface in $S^3$, the expression $1 - K$ is a so called absolute value type function [5], the absolute value of a holomorphic function (which may have zeros) multiplied by a positive function. Then $\Delta \log(1 - K)$ is still defined at the zeros of $1 - K$.

\(^2\)The term “curve” means complex curve, parametrized on a Riemann surface.
$S = S^2$ these are the meromorphic functions on $M$. In the present paper we are dealing with the other case $S = S^6$. In particular, these maps are conformal and harmonic, hence (possibly branched) minimal immersions.

The subject was started by Bryant [3] who described pseudoholomorphic curves in terms of an adapted frame, called Frenet frame in analogy to real curves in 3-space, and he gave examples for pseudoholomorphic curves on compact Riemann surfaces of any genus. Bolton, Vrancken and Woodward [2] characterized pseudoholomorphic curves among the minimal surfaces in $S^6$. The intrinsic characterization (46) was given by Hashimoto [9]. In the present paper, we will use the same Frenet frame but our method is different from that of [3, 9]. Instead of the Maurer-Cartan equation we use an embedding theorem [6] into reductive homogeneous spaces which reduces the a-priori number of integrability conditions considerably.

In section 2 we briefly describe our method. After recalling the necessary background on octonionic computations and pseudoholomorphic maps in the 6-sphere (sections 3 - 7), we derive in section 10 the equations for the Frenet frame in terms of the canonical connection introduced in sections 8, 9. The main results are stated and proved in sections 12 for $S^6$ and in 13 for $S^5$ (a case which was not treated by Hashimoto [9]). We try to give complete computations with all details.

2. Embedding into a homogeneous space

Let $S = G/H$ be a Riemannian homogeneous space. If a smooth map $f : M \to G/H$ is given, there exists locally a smooth “lift” $F : M \to G$ with $f = \pi \circ F$ where $\pi : G \to G/H$ is the canonical projection. Choosing a basis $b = (b_1, \ldots, b_k)$ of the tangent space $T_{p_0}S$ where $p_0 = eH \in G/H$ is the base point in $S$, we may consider $F$ as the “moving frame” $Fb = (F_1, \ldots, F_k)$ where each $F_j = Fb_j$ is a vector field along the map $f$. The lift $F$ in turn can be described by the $g$-valued one-form $\alpha = F^{-1}dF$. Vice versa, if an arbitrary $g$-valued one-form $\alpha$ on a simply connected manifold $M$ is given, we look for a map $F : M \to G$ with

$$dF = F\alpha.$$  

This is an overdetermined system, and the local existence of solutions is equivalent to an integrability condition for the coefficient matrix $\alpha$, the Maurer-Cartan equation $d\alpha = [\alpha, \alpha]$.

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3To simplify notation, we think of $G$ as a matrix group $G \subset \mathbb{R}^{n \times n}$. 
In the present paper we replace (1) by the equation
\[ \nabla F = F\beta \]
where \( \nabla \) is a canonical \( G \)-connection on \( S = G/H \) (holonomy in \( G \) and parallel curvature and torsion).\(^4\) Now \( \beta \) is a one-form on \( M \) which takes values in \( \mathfrak{h} \) rather than in \( \mathfrak{g} \). Local existence of a solution \( F \) of (2) is still equivalent to an integrability condition, but the size of the coefficient matrix \( \beta \) is considerably smaller. In the present case of pseudoholomorphic curves in \( \mathbb{R}^6 \), the integrability condition for \( \beta \) is
\[ [B', B''] + (B'')_x - (B')_x = \text{diag}(\lambda, -\lambda/2, -\lambda/2) \]
(see section 10), where \( \lambda \) is the conformal factor and \( \beta = B'dz + B''d\bar{z} \). This is due to an embedding theorem for homogeneous spaces in [6].

Our frame \( F \) along a pseudoholomorphic curve \( f \) in \( \mathbb{S}^6 \) is that of Bryant and Hashimoto [3, 9], up to ordering. It is an adapted frame which take care of the higher normal spaces of the immersion. In particular, \( F_1 \) is essentially the differential of \( f \) itself.

3. Octonions

A finite dimensional algebra \( \mathbb{A} \) over \( \mathbb{R} \) with euclidean inner product is called “normed” if \( |ab| = |a||b| \) for any \( a, b \in \mathbb{A} \). We have an orthogonal decomposition \( \mathbb{A} = \mathbb{R} \cdot 1 \oplus \mathbb{A}' \) where \( \mathbb{A}' \) is called the space of imaginary elements of \( \mathbb{A} \). Every nonzero \( a \in \mathbb{A} \) has an inverse \( a^{-1} = \bar{a}/|a|^2 \)
where \( \bar{a} = a_o - a' \) for \( a = a_o + a' \) with \( a_o \in \mathbb{R} \) and \( a' \in \mathbb{A}' \). There are only four normed algebras: \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \) (real and complex numbers, quaternions and octonions), and the octonions \( \mathbb{O} \cong \mathbb{R}^8 \) contain all the others. Octonions are not associative, but still computations are easy if one observes the following three rules which follow almost immediately from the equation \( |ab| = |a||b| \):\(^5\)

1. Any unit vector \( a \in \mathbb{O}' \) generates a subalgebra isomorphic to \( \mathbb{C} \) where \( a \) plays the role of \( i \).

\(^4\)More precisely, a connection \( \nabla \) on \( TS \) is called \( G \)-connection if all its parallel displacements are given by elements of \( G \). Then the one-form \( \beta \) in (2) takes values in \( \mathfrak{h} \). In fact, along any curve \( c: (-\epsilon, \epsilon) \to S \) starting at the base point \( p_0 = eH \in S \) we have \( \nabla_c F_v = \partial_v(\tau^{-1} F) \) for any \( v \in T_{p_0} S \), where \( \tau \) is the parallel transport along \( c \). Since \( \tau^{-1} F \) takes values in \( G \) and fixes \( p_0 \), it takes values in \( H \), hence \( \nabla_c F \in \mathfrak{h} \).

\(^5\)If \( a \in \mathbb{O}' \) and \( |a| = 1 \), then \( |1 + a| = \sqrt{2} \), hence \( |1 + a||1 - a| = 2 \). On the other hand, \( (1 + a)((1 - a)x) = (1 - a)x + a(x - ax) = x - a(ax) \) for all \( x \in \mathbb{O} \), and \( |(1 + a)((1 - a)x)| = 2|x| \). This is impossible unless the two vectors \( x \) and \( -a(ax) \) (which have equal length) are equal, \( a(ax) = -x \). This shows (1), and (2), (3) can be proved similarly.
(2) Any two orthonormal \( a, b \in \mathbb{O}' \) generate a subalgebra isomorphic to \( \mathbb{H} \) where \( a, b, ab \) play the rôles of \( i, j, k \); they are associative and anti-commutative, \( ab = -ba \).

(3) Any three orthonormal \( a, b, c \in \mathbb{O}' \) with \( c \perp ab \) (“normed Cayley triples”) generate the algebra \( \mathbb{O} \); they are anti-associative, \( a(bc) = -(ab)c \).

Let \( 1, i, j, k, l, il, jl, kl \) be the standard basis of \( \mathbb{O} = \mathbb{H} + \mathbb{H}l \). Then \((i, j, l)\) is a normed Cayley triple, and so is its image \((\alpha i, \alpha j, \alpha l)\) under any automorphism \( \alpha \) of \( \mathbb{O} \); note that \( \alpha \) is orthogonal.\(^6\) Vice versa, given any normed Cayley triple \((a, b, c)\), there is precisely one automorphism \( \alpha \) of \( \mathbb{O} \) with \( a = \alpha i, b = \alpha j, c = \alpha l \). Thus the space of normed Cayley triples is a manifold of dimension \( 6 + 5 + 3 = 14 \) on which the exceptional group \( G_2 = \text{Aut}(\mathbb{O}) \subset SO_7 \) acts simply transitively. In particular, \( G_2 \) acts transitively on \( S^6 \).

We will also need the complexified octonions \( \mathbb{O}_c = \mathbb{O} \otimes \mathbb{C} = \mathbb{O} \oplus i\mathbb{O} \) (we distinguish \( i = \sqrt{-1} \) from \( i \in \mathbb{O} \)). This is no longer a division algebra: there are zero devisors, e.g. \( 1 + ia \) for any \( a \in S^6 \subset \mathbb{O}' \). However, analytic formulas which hold in \( \mathbb{O} \) extend to \( \mathbb{O}_c \). E.g. for \( a \in \mathbb{O}' \) and \( b \in \mathbb{O} \) we have (using rule (2))

\[
(a(ba)) = a^2b = -(a,a)b,
\]

and this remains true for \( a \in \mathbb{O}_c', b \in \mathbb{O}_c \) where \( \langle , \rangle \) is the complexified inner product. In particular \( a(ab) = 0 \) when \( \langle a, a \rangle = 0 \). Other useful formulas which extend for all \( a, b, c \in \mathbb{O}_c \) are

\[
\langle ab, ac \rangle = \langle a, a \rangle \langle b, c \rangle
\]

and the antisymmetry of \( \langle ab, c \rangle \) in all three variables.

As \( \mathbb{O} \) is decomposed into planes that are invariant under left multiplication with \( \mathbb{C} \subset \mathbb{O} \), we may decompose \( \mathbb{O}_c \) into free \( \mathbb{C}_c \)-modules where \( \mathbb{C}_c = \mathbb{C} \otimes_R \mathbb{C} \) is the complexification of \( \mathbb{C} \). A complex Cayley triple is a triple \((a, b, c)\) in \( \mathbb{O}_c' \) where \( a \) lies in \( \mathbb{C}_c \) (or in an isomorphic subalgebra) and where \( b, c \) belong to two perpendicular \( \mathbb{C}_c \)-modules. Like its real analogue, a complex Cayley triple is anti-associative, \((ab)c = -(a(bc))\).

4. The nearly Kähler structure on \( S^6 \)

The 6-sphere \( S^6 \) plays a similar rôle for the octonions \( \mathbb{O} \) as the 2-sphere \( S^2 \) for the quaternions \( \mathbb{H} \): they are unit spheres in \( A' \), the imaginary part of the division algebra \( A = \mathbb{O}, \mathbb{H}, \) respectively, a fact which

\(^6\)Any automorphism of \( \mathbb{O} \) is orthogonal: it preserves real and imaginary octonions since real octonions are real multiples of 1 and imaginary octonions are those which square to negative real multiples of 1. Thus an automorphism preserves the conjugation \( a^* = \text{Re} \ a - \text{Im} \ a \) and also the norm \( |a|^2 = a^*a \) for any \( a \in \mathbb{O} \).
has been observed and applied in [8]. Each \( p \in \mathbb{S} \) satisfies \((L_p)^2 = -I\) where \( L_p : x \mapsto px \) denotes the left multiplication with \( p \). Hence \( L_p \) is a complex structure preserving the plane \( \text{Span} \{1, p\} \) and its orthogonal complement, the tangent space \( T_p \mathbb{S} \). Thus \( J_p := L_p|T_p \mathbb{S} \) is a complex structure on \( T_p \mathbb{S} \) and defines an almost complex structure \( J \) on \( \mathbb{S} \). It is convenient to use the cross product \( a \times b \) which is the imaginary \((A')-\) part of the product \( ab \) for any \( a, b \in A' \):

\[
a \times b = (ab)' = \begin{cases} ab & \text{when } a \perp b, \\ 0 & \text{when } a, b \text{ lin. dependent} \end{cases}
\]

Then each \( J_p \) extends to a linear map on \( A' \),

\[
(5) \quad J_p(v) = p \times v,
\]

and the derivative of the matrix-valued linear map \( J : A' \to \text{End}(A') : p \mapsto J_p \) is \((\partial_v J)w = v \times w\). Denoting by \( D = \partial^T \) the Levi-Civita derivative on \( \mathbb{S} \), we have

\[
(6) \quad (D_v J)w = (v \times w)_{p^\bot} = v \times w - \langle v \times w, p \rangle p
\]

where \( p \in \mathbb{S} \) is the position vector and \( v, w \in T_p \mathbb{S} = p^\perp \). In particular \((\partial_v J)v = v \times v = 0 \) and therefore

\[
(7) \quad (D_v J)v = 0.
\]

A Riemannian manifold with an almost complex structure \( J \) with this property is called nearly Kähler.\(^7\)

An orthogonal linear map \( g \) on \( O' \) which preserves the almost complex structure \( J \) satisfies \( gJ_p(v) = J_{gp}(gv) \) for any \( p, v \in O' \) with \( v \perp p \). By \((5)\) this is equivalent to \( g(pv) = (gp)(gv) \) which holds if and only if \( g \in G_2 = \text{Aut}(O') \subset SO_7 \). Thus \( G_2 \) is precisely the group of isometries \( g \) on \( S^6 \) which are pseudoholomorphic, that is their differentials \( dg_p : T_p S^6 \to T_{gp} S^6 \) are complex linear with respect to the complex structures given by \( J \) on the tangent spaces of \( S^6 \). The stabilizer subgroup \( H = (G_2)_p \) of any \( p \in S^6 \) (say: \( p = l \)) preserves the tangent space \( T_p S^6 \) and its complex structure \( J_p \), making \( T_p S^6 \) a 3-dimensional complex vector space. Identifying \((T_p S^3, J_p)\) with \( \mathbb{C}^3 \) we obtain \( H \subset U_3 \).

But \( H \) preserves also the antisymmetric 3-form \( \langle vw, w \rangle \) on \( T_p S^6 \) which can be viewed as the real part of a complex determinant, thus \( H \subset SU_3 \), and by dimension reasons we have equality \( H = SU_3 \).

\(^7\)In the case of \( S^2 \) we even obtain \( DJ = 0 \) (Kähler property) since \( v \times w \) is normal when \( v, w \) are tangent vectors, hence \((D_v J)w = (v \times w)^T = 0\).
5. Pseudoholomorphic curves

Let $M$ be a Riemann surface. A smooth map $f: M \to \mathbb{S}^6$ is called pseudoholomorphic if it is holomorphic with respect to this almost complex structure $J_f v = p \times v$. In other words, if $z = x + iy$ is a conformal coordinate on $M$, the corresponding partial derivatives $f_x, f_y$ satisfy

$$f \times f_x = f_y, \quad f \times f_y = -f_x. \tag{8}$$

Clearly, such map is conformal since $|f_x| = |f_y|$ and $f_x \perp f_y$. Further $f$ is harmonic, that is $f_{xx} + f_{yy}$ is a normal vector, a multiple of $f$. In fact, differentiating (8) we obtain

$$f_{yy} = (f \times f_y)_y = f_y \times f_x + f \times f_{xy}, \quad f_{xx} = -(f \times f_y)_x = -f_x \times f_y - f \times f_{yx}, \tag{9}$$

and hence

$$f_{yy} + f_{xx} = 2f_y \times f_x, \quad f_{yy} - f_{xx} = 2f \times f_{xy}. \tag{10} \tag{11}$$

The first equation (10) shows that $f$ is harmonic: $f_y \times f_x$ is proportional to $f$ since by (8), $f, f_x, f_y$ span a quaternion subalgebra. Moreover

$$f_{yx} = (f \times f_x)_x = f \times f_{xx} = Jf_{xx}. \tag{12}$$

It is convenient to use the complex derivatives $f_z = \frac{1}{2}(f_x - if_y)$ and $f_{zz} = \frac{1}{4}(f_z - if_y)_z - i(f_x - if_y)_y = \frac{1}{4}(f_{xx} - f_{yy} - 2i f_{xy}) = \frac{1}{4}(J + i)f_{xy}$. Hence

$$f_z = -(J + i)f_y/2, \quad f_{zz} = -(J + i)f_{xy}/2. \tag{13}$$

Since $(J - i)(J + i) = 0$, these vectors belong to the $i$-eigenspace $E_z$ of $J_f: v \mapsto f \times v$ on $T_f \mathbb{S}^6$. This is an isotropic subspace, i.e. $\langle v, v \rangle = 0$ for all $v \in E_z$: If $v = (J + i)a$, then $\langle v, v \rangle = \langle Ja, Ja \rangle - \langle a, a \rangle + 2i\langle Ja, a \rangle = 0$.

**Lemma 5.1.** Putting $\lambda = \langle f_z, f_z \rangle = |f_z|^2$ and $l = \log \lambda$, we have

$$f_{zz} = f_{zz}^\perp + l_z f_z, \quad (f_z^\perp)^\perp = -\lambda f_z, \quad (f_{zz}^\perp)^\perp = -(\lambda + l_z z) f_z. \tag{14}$$

**Proof.** To prove the first equation we note $\langle f_{zz}, f_z \rangle = \frac{1}{2} \langle f_z, f_z \rangle = 0$ and $\langle f_{zz}, f_z \rangle = \lambda_z - \langle f_z, f_z \rangle = \lambda_z$ since $\langle f_z, f_z \rangle = \frac{1}{2} \langle f_z, f_z \rangle = 0$. Hence $f_{zz} - f_{zz} = f_{zz}^\perp = \frac{1}{2} \langle f_{zz}, f_z \rangle = (\lambda_z / \lambda) f_z = l_z f_z$. The second equation follows since $4f_{zz} = (f_z - if_y)_x + i(f_x - if_y)_y = f_{xx} + f_{yy}$, and this is a multiple of $f$. To determine the multiple we
we compute the inner product $\langle f_z \bar{z}, f \rangle = \langle f_z, f \rangle \bar{z} - \langle f_z, f \rangle = -\lambda$ since $\langle f_z, f \rangle = \frac{1}{2} \langle f, f \rangle = 0$. This shows the second equality.

The third equality follows from the two previous ones: From $f^\perp_{zz} = f_{zz} - l_z f_z$, we have $\langle f_{zz}^\perp \rangle = f_{zz} - (l_z f_z)\bar{z} = -(\lambda f)\bar{z} - l_z f_z + l_z \lambda f = -l_z f_z$, using $\lambda = l_z \lambda$.

As a consequence, $f_z$ and $f_{zz}^\perp$ are holomorphic sections of the complexified tangent and normal bundles $T^c$ and $N^c$ of $f : M \to S$, since $(f_z)\bar{z}$ and $(f_{zz}^\perp)\bar{z}$ have zero projection to $T^c$ and $N^c$, respectively. Thus the isotropic subbundles $T' = \mathbb{C} f_z$ and $N'_1 = \mathbb{C} f_{zz}^\perp$ are well determined even at possible zeros of these sections, and by isotropy the same holds for the real bundles $T$ and $N_1$, the tangent bundle and the first normal bundle of $f$. Hence along $f$, the tangent bundle of $S^6$ splits into three $J$-invariant orthogonal plane bundles, $f^*(TS^6) = T \oplus N_1 \oplus N_2$.

The full $(+i)$-eigenspace $E_+ = T^i_f S^6$ is spanned by

\begin{equation}
\begin{align*}
F_1 &= f_z, \\
F_2 &= f_{zz}^\perp, \\
F_3 &= \overline{F_1 F_2} = f \times f_{zz}. 
\end{align*}
\end{equation}

The third line follows since $(f, F_1, F_2)$ is a complex Cayley triple, hence $f(F_1 F_2) = -(f F_1) F_2 = -i F_1 F_2$ and therefore $F_1 F_2 = \mathbb{C} F_2$ is an (unnormed) Cayley triple, hence $(f, F_1, F_2)$ is a conformal Frenet frame.

The three vectors $F_1, F_2, F_3$ together with their complex conjugates $\overline{F_1}, \overline{F_2}, \overline{F_3}$ form bases of the complexified bundles $T^c, N'_1, N'_2$, respectively, and the only nonzero inner products are

\begin{equation}
\langle F_1, \overline{F_1} \rangle = \lambda, \quad \langle F_2, \overline{F_2} \rangle = \mu, \quad \langle F_3, \overline{F_3} \rangle = 2\lambda \mu.
\end{equation}

The last equality is seen as follows: If $F_1 = (f + i) a$ and $F_2 = (f + i) b$, then $F_1 F_2 = ((fa + ia)(fb + ib)) = (fa)(fb) - ab + i((fa)b + a(fb))$. If $(f, a, b)$ is an (unnormalized) Frenet frame, then so is $(f, f a, f b)$, and $(fa)(fb) = ((fa)f)b = -ab$ (using $|f| = 1$) while $a(fb) = -(a)b = (fa)b$. Thus $F_1 F_2 = -2ab + 2i(fa)b$, and $|F_1 F_2|^2 = 8|a|^2|b|^2$ while $|F_1|^2 |F_2|^2 = 4|a|^2|b|^2$.

\textbf{Remark 1.} Later we will also use the normalized Frenet frame

\begin{equation}
\begin{align*}
F_1' &= F_1 / \sqrt{\lambda}, \quad F_2' = F_2 / \sqrt{\mu}, \quad F_3' = F_3 / \sqrt{2 \lambda \mu}.
\end{align*}
\end{equation}

\textbf{Corollary 5.1.} Let $f : M \to S^6$ pseudoholomorphic and $z$ a conformal coordinate on $M$. Then $\mu = |f_z|^2$ depends on $\lambda = |f_z|^2$:

\begin{equation}
\mu = \lambda^2 (1 - K) = \lambda(\lambda + l_z) \quad \text{where} \quad l = \log \lambda.
\end{equation}
The higher fundamental forms $v$ for arbitrary tangent vectors $z$ conformal minimal immersion $f$ of all mixed components of $A$ and $(\ )^{N}$ induces metric $s$ for $f$.

Further we have used (13) to see

The first equality in (18) follows since the Gaussian curvature $K$ of the manifold $ds^2 = 2\lambda \cdot dz \, d\bar{z}$ on $M$ satisfies

\[ \lambda K = -(\log \lambda)_{z\bar{z}} = -l_{z\bar{z}}, \]

thus $\lambda(1 - K) = \lambda + l_{z\bar{z}}$.

\[ \lambda = 2|f_{x}|^2 = |f_{x}|^2 = |f_{y}|^2, \]

\[ 2\mu = 2|f_{z}|^2 = |f_{z}|^2 = |f_{y}|^2. \]

6. The generalized Hopf differentials

For any conformal harmonic map $f : M \to \mathbb{S}^n$ on a Riemann surface $M$ one considers the higher fundamental forms

\[ A_{k}(v_1, \ldots, v_k) = (\partial_{v_1} \cdots \partial_{v_k} f)^{N_{k-1}} \]

for arbitrary tangent vectors $v_1, \ldots, v_k$, where $N_0 = T$ is the tangent space and $N_{k-1}$ for $k \geq 2$ the $(k-1)$-th normal space\footnote{Putting $E_{k}$ the span of all derivatives of $f$ with degree up to $k$ where $k \geq 2$, we define $N_{k-1}$ recursively as the orthogonal complement of $N_{k-2}$ in $E_{k}$, where $N_0$ is the tangent space, the span of the first derivatives.} on the surface. Using a conformal coordinate $z$ on $M$, the harmonicity of $f$ yields the vanishing of all mixed components of $A_{k}$ (those involving both $dz$ and $d\bar{z}$). Thus

\[ A_{k} = B_{k} + \overline{B}_{k} \quad \text{with} \quad B_{k} = \left( \left( \frac{\partial}{\partial z} \right)^{k} f \right)^{N_{k-1}} dz^{k}, \]

see [12] for details. The generalized Hopf differential is the symmetric $2k$-form on $M$ defined by

\[ \Lambda_{k} = \langle B_{k}, B_{k} \rangle. \]

The first Hopf differential $\Lambda_{1} = \langle f_{z}, f_{\bar{z}} \rangle dz^{2}$ vanishes by conformality of $f$, and the second one $\Lambda_{2} = \langle f_{zz}, f_{\bar{z}\bar{z}} \rangle dz^{4}$ is the classical Hopf differential which is holomorphic for every conformal harmonic map. More generally, $\Lambda_{k}$ is holomorphic if $\Lambda_{1}, \ldots, \Lambda_{k-1}$ vanish everywhere, cf. [12].

\[ \text{Proof.} \quad \text{From} \quad \langle f_{zz}^{\perp}, f_{\bar{z}} \rangle = 0 \quad \text{we obtain using the third equation of (14):} \]

\[ 0 = \langle f_{zz}^{\perp}, f_{\bar{z}} \rangle_{\bar{z}} = -(\lambda + l_{z\bar{z}}) \langle f_{z}, f_{\bar{z}} \rangle + \langle f_{zz}^{\perp}, f_{\bar{z}} \rangle = -(\lambda + l_{z\bar{z}})\lambda + \mu. \]

Remark 2. Equation (18) is just the Gauss equation (G) for the conformal minimal immersion $f : M \to \mathbb{S}^6$:

\[ 4\lambda^2 (K - 1) = |f_{x}|^2 |f_{y}|^2 (K - 1) \quad \overset{(G)}{=} \quad \langle f_{xx}^{\perp}, f_{yy}^{\perp} \rangle - |f_{xy}^{\perp}|^2 = -2|f_{xx}^{\perp}|^2 = -4\mu. \]

For “$\overline{\mu}$” recall that $f_{yy}^{\perp} = -f_{xx}^{\perp}$ (harmonicity) and $f_{xy} = Jf_{xx}$ (see (12)). Further we have used (13) to see

\[ 2\lambda = 2|f_{x}|^2 = |f_{x}|^2 = |f_{y}|^2, \]

\[ 2\mu = 2|f_{z}|^2 = |f_{z}|^2 = |f_{y}|^2. \]
If $M$ is compact of genus 0, all holomorphic differentials vanish, hence all $\Lambda_k$ are zero. This is the superminimal case investigated first by Calabi [4].

In our case of pseudoholomorphic maps $f : M \to \mathbb{S}^6$, we have always $\Lambda_2 = 0$ since $f_{zz}$ lies in the isotropic space $E_\perp$. Therefore $\Lambda_3 = \langle f_{zzz}, f_{zz} \rangle dz^6$ is holomorphic. For completeness and to introduce notation we give a direct proof.

**Lemma 6.1.** Let $f : M \to \mathbb{S}^6$ a pseudoholomorphic curve and $z$ a conformal coordinate on $M$. Then the function $h := \langle f_{zzz}, f_{zz} \rangle$ is holomorphic with

$$h = \langle f_{zzz}, f_{zz} \rangle = \langle (F_2)_z, (F_2)_z \rangle = \langle (F_2)_z^N, (F_2)_z^N \rangle,$$

and $\Lambda_3 = h(z) dz^6$.

**Proof.** $\langle f_{zzz}, f_{zz} \rangle z = 2\langle f_{zzzz}, f_{zz} \rangle = -2\langle (\lambda f)_{zz}, f_{zz} \rangle = 0$ since $f_{zz}$ is perpendicular to $f, f_{zz}$. In fact, $\langle f, f_{zz} \rangle = \langle f, f_{zz} \rangle_z = 0$ since $\langle f, f_{zz} \rangle = -\langle f_z, f_z \rangle = 0$, further $\langle f_z, f_{zz} \rangle = -\langle f_{zz}, f_{zz} \rangle = 0$ and $\langle f_{zz}, f_{zzz} \rangle = \frac{1}{2} \langle f_{zz}, f_{zz} \rangle_z = 0$. Thus $h$ is holomorphic and $h(z) dz^6$ defines a holomorphic 6-form on $M$.

From (14) we have $f_{zz} = F_2 + l_z f_z$, and thus $\langle f_{zz} - F_2 \rangle_z = l_z f_z$ belongs to the span of $f_z$ and $f_{zz}$ which is part of the isotropic subspace $E_\perp$. Further, since $f_{zzzz} \perp f, f_{zz}$, we have $f_{zzzz} - f_{zzzz} \in \text{Span} (f_z, f_{zz})$. (The components of $f_{zzzz}$ proportional to $f_z, f_{zz}$ involve the inner products with $f_z, f_{zz}$ which are zero.) Thus $h = \langle f_{zzz}, f_{zz} \rangle = \langle (F_2)_z, (F_2)_z \rangle = \langle (F_2)_z^N, (F_2)_z^N \rangle$, and $h(z) dz^6 = \Lambda_3$. Moreover, $(F_2)_z \perp f, F_1, F_2$, hence $(F_2)_z = (F_2)_z^N \in \text{Span} \{F_1, F_2\}$, and this component does not contribute to the inner product $\langle (F_2)_z, (F_2)_z \rangle$. This proves the last equality in (19).

7. The derivatives of the Frenet frame

**Proposition 7.1.** Let $f : M \to \mathbb{S}^6$ be a pseudoholomorphic curve with Frenet frame $F_1, F_2, F_3$ as in (15), corresponding to a conformal coordinate $z$ on $M$. Let $\lambda = |F_1|^2$, $\mu = |F_2|^2$ and $l = \log \lambda$, $m = \log \mu$. Then:

---

9A conformal harmonic map $f : M \to \mathbb{S}^{2m}$ with all $\Lambda_k = 0$ but the highest one $\Lambda_{m-1}$ (which then must be holomorphic) is called superconformal.
\[(F_1)_z = l_z F_1 + F_2, \]
\[(F_2)_z = m_z F_2 + (ih/2\lambda \mu) F_3 - (i/2)\bar{F}_3, \]
\[(F_3)_z = i\lambda \bar{F}_2, \]
\[(F_1)_\bar{z} = -\lambda f, \]
\[(F_2)_\bar{z} = -\mu F_1, \]
\[(F_3)_\bar{z} = (ih/\mu) F_2 + (l_\bar{z} + m_\bar{z}) F_3. \]

**Proof.** The equations for \((F_1)_z, (F_1)_\bar{z}\) and \((F_2)_\bar{z}\) follow directly from (14) using \(\lambda + l_\bar{z} = \mu/\lambda\), see (18). The equation for \((F_3)_\bar{z} = (f_z \times f_{zz})_\bar{z}\) is proved as follows:

\[
(f_z \times f_{zz})_\bar{z} = f_{z\bar{z}} \times f_{zz}^+ + f_z \times (f_{zz}^+)_\bar{z} \quad \overset{(14)}{=} \quad -\lambda f \times f_{zz}^+ - \mu f_z \times f_z \quad = \quad -i\lambda f_z^+. \]

The equations for \((F_2)_\bar{z}\) and \((F_3)_\bar{z}\) are proved in the subsequent two lemmas. \(\Box\)

**Lemma 7.1.**

\[(f_{zz}^+)_z = m_z f_{zz}^+ + ih/(2\lambda \mu) f_z \times f_{z\bar{z}} - (i/2) f_z \times f_{zz}, \]

where \(l = \log \lambda\) and \(m = \log \mu\).

**Proof.**

\[
\langle (f_{zz}^+)_z, f_z \rangle = -\langle f_{zz}^+, f_{zz} \rangle = 0, \quad (a) \\
\langle (f_{zz}^+)_z, f_{\bar{z}} \rangle = -\langle f_{z\bar{z}}^+, f_{zz} \rangle = \langle f_{zz}^+ \lambda f_z \rangle = 0, \quad (b) \\
\langle (f_{zz}^+)_z, f_{\bar{z}z}^+ \rangle = (1/2) \langle f_{zz}^+, f_{zz}^+ \rangle = 0, \quad (c) \\
\langle (f_{zz}^+)_z, f_{\bar{z}z} \rangle = \langle f_{z\bar{z}}^+, f_{\bar{z}z} \rangle + \langle f_{z\bar{z}}^+, (\lambda f \bar{z})_\bar{z} \rangle = \mu_z, \quad (d) \\
\langle (f_{zz}^+)_z, f_z \times f_{\bar{z}z}^+ \rangle = \langle f_{zz}^+ \lambda f_z f_{\bar{z}z} + f_z \times (\lambda f \bar{z})_\bar{z} \rangle = -i\lambda \mu. \quad (e) \\
\]

The last equation \((e)\) tells us

\[
\langle (F_2)_z, F_3 \rangle = -i\lambda \mu. \quad (e) \\
\]

It remains to compute \(\langle (F_2)_z, \bar{F}_3 \rangle\), using

\[h = \langle (F_2)_z^{N_2}, (F_2)_\bar{z}^{N_2} \rangle. \]

We have

\[2\lambda \mu (F_2)_z^{N_2} = \langle (F_2)_z, \bar{F}_3 \rangle F_3 + \langle (F_2)_z, F_3 \rangle \bar{F}_3 \]

and hence

\[(2\lambda \mu)^2 h = 2 \langle (F_2)_z, \bar{F}_3 \rangle \cdot \langle (F_2)_z, F_3 \rangle \cdot \langle F_3, \bar{F}_3 \rangle \]
\[= 2 \langle (F_2)_z, \bar{F}_3 \rangle \cdot (-i\lambda \mu) \cdot 2\lambda \mu \]
Thus we obtain:

\[ a \langle F_2, F_2 \rangle = \langle (F_2)_z, F_3 \rangle = \mu_z, \quad (d) \]
\[ b \langle F_3, F_3 \rangle = \langle (F_2)_z, F_3 \rangle = \ii \hbar, \quad (f) \]
\[ c \langle F_3, F_3 \rangle = \langle (F_2)_z, F_3 \rangle = -i\lambda \mu. \quad (e) \]

Thus

\[ a = \mu_z/\mu = m_z, \]
\[ b = \ii \hbar/(2\lambda \mu), \]
\[ c = -i\lambda \mu/(2\lambda \mu) = -i/2. \]

Lemma 7.2.

\[ (f_z \times f_{zz})_z = -(\ii \hbar/\mu) f_z^+ + (l_z + m_z) f_z \times f_{zz}. \]

Proof. We compute the components of \((f_z \times f_{zz})_z\). Using \(f_z \times f_{zz} \in N_2^c \perp \mathbb{T}^c \oplus N_1^c\), we obtain:

\[
\begin{align*}
\langle (f_z \times f_{zz}), f_z \rangle &= -\langle f_z \times f_{zz}, f_z \rangle = 0, \\
\langle (f_z \times f_{zz}), f_z \rangle &= \langle f_z \times f_{zz}, \lambda f_z \rangle = 0, \\
\langle (f_z \times f_{zz}), f_z^+ \rangle &= -\langle f_z \times f_{zz}, (f_z^+) \rangle = -\ii \hbar, \quad (21) \\
\langle (f_z \times f_{zz}), f_{zz}^+ \rangle &= \langle f_z \times f_{zz}, (\lambda f_z) \rangle = 0, \\
\langle (f_z \times f_{zz}), f_z \times f_{zz} \rangle &= \langle f_z \times f_{zz}, f_z \times f_{zz} \rangle = 0, \\
\langle (f_z \times f_{zz}), f_z \times f_{zz} \rangle &= \langle f_z \times f_{zz}, f_z \times f_{zz} \rangle = 2(\lambda \mu)_z.
\end{align*}
\]

where \(\mathbb{\oplus}\) follows since \((f_z \times f_{zz})_z = i\lambda f_z^+ \perp N_2\). Thus we obtain \((f_z \times f_{zz})_z = a f_z^+ + b f_z \times f_{zz}\) with

\[
\begin{align*}
a \cdot \mu &= \langle (f_z \times f_{zz})_z, f_z^+ \rangle = -\ii \hbar, \\
b \cdot 2\lambda \mu &= \langle (f_z \times f_{zz})_z, f_z \times f_{zz} \rangle = 2(\lambda \mu)_z
\end{align*}
\]

which shows \(a = -\ii \hbar/\mu\) and \(b = \log(\lambda \mu)_z = l_z + m_z\). \qed
8. The canonical $G_2$ connection

The three vectors $F_1 = f_z$, $F_2 = f_{zz}$, $F_3 = f_z \times f_{zz}$ defined in (15) (spanning the isotropic subspace $E_+ = \{ v \in \mathbb{C}^3 : f \times v = iv \}$) are positive real multiples of $i-iil$, $j-ilj$, $k-ilk$, up to transformation with some element of $G_2 = \text{Aut}(\mathbb{C})$. Thus, up to positive factors, $F = (F_1, F_2, F_3)$ can be considered as a moving $G_2$-frame, a section of the $SU_3$-principal bundle $G_2 \to G_2/SU_3 = S^6$, pulled back to $M$ via $f$. But as we see from Proposition 7.1, the derivative $DF$ cannot be expressed in terms of $F$ alone; one also needs $F$. The reason is that covariant derivatives on $S^6$ relies on the Levi-Civita parallel displacements which unfortunately does not preserve $J$, it is not in $G_2$. Therefore we will use another connection $\nabla$ on $S^6$, whose parallel displacements belong to $G_2$, a $G_2$-connection or hermitian connection. Thus we will derive formulas of the type $\nabla F = FB'$ and $\nabla F = FB''$ for some complex $3 \times 3$-matrices $B', B''$. It turns out that $B', B''$ depend only on the metric coefficients of the surface $f$ and some given holomorphic 6-form $\Lambda$; this will prove existence and uniqueness of pseudoholomorphic maps.

A $G_2$-connection $\nabla = D + A$ needs to make $J$ parallel,

$$0 = \nabla_v J = D_v J + [A_v, J]$$

where $(D_v J)w = (v \times w)_p$ for $v, w \in T_p S^6 = p^\perp$. Thus $[A_v, J] = -D_v J$.

We may split $A_v = A_v^+ + A_v^-$ where $A_v^+$ commutes with $J$ and $A_v^-$ anticommutes with $J$. Then $-D_v J = [A_v, J] = [A_v^-, J] = 2A_v^-, J$, hence $A_v^- = \frac{1}{2}(D_v J)J$ while $A_v^+$ is unrestricted.

Among the $G_2$-connections there is the canonical connection (see also [1]) which has the additional property that $G_2$ acts on $S^6$ by affine transformations: $\nabla gV(gW) = g(\nabla V W)$ for any $g \in G_2$ and any two tangent vector fields $V, W$ on $S^6$. Clearly $G_2 \subset SO_7$ is affine also for the Levi-Civita connection $D$, hence it keeps $A = \nabla - D$ invariant. In particular, fixing a base point $p \in S^6$, say $p = l$, the tensor $A$ at $p$ is invariant under the isotropy group $SU_3$ at $l$. Thus the map

$$v \mapsto A_v^+ : T_p S^6 = \mathbb{C}^3 \to \mathbb{C}^{3 \times 3}$$

is $SU_3$-equivariant. The group $SU_3$ acts on the matrix space $\mathbb{C}^{3 \times 3}$ by conjugation, splitting it into two equivalent subrepresentations (hermitian and antihermitian matrices) both of which are irreducible up to a one-dimensional fixed space. Thus there is no nonzero equivariant linear map $\mathbb{C}^3 \to \mathbb{C}^{3 \times 3}$. Therefore the canonical connection satisfies $A_v^+ = 0$, hence $A_v = A_v^- = \frac{1}{2}(D_v J)J$ and therefore

$$\nabla_v J = D_v J + A_v, \quad 2A_v = (D_v J)J.$$

Now $\nabla_v J = [\nabla_v, J] = [D_v, J] + [A_v, J] = 0$. 

9. Canonical torsion and curvature on $\mathbb{S}^6$

It is well known that a canonical connection has parallel torsion and curvature tensors which we are going to compute now. Let us put

\begin{equation}
S_v = D_v J .
\end{equation}

Since $J_p v = p \times v$ for any $p \in \mathbb{S}^6$ and $v \in T_p \mathbb{S}^6 = p^\perp$, we have $S_v w = (D_v J) w = (v \times w)_{p^\perp} = v \times w - \langle p, v \times w \rangle p$, and since $\langle p, v \times w \rangle = \langle p, v \times w \rangle = \langle Jv, w \rangle$, we obtain

\begin{equation}
S_v w = v \times w - \langle Jv, w \rangle p = (vw)_{p^\perp}.
\end{equation}

where $p$ is the position vector, $v, w \in T_p \mathbb{S}^6$, and $( )_{p^\perp}$ denotes the projection onto $T_p \mathbb{S}$. Using the fact that the parallel displacements of $\nabla$ belong to the group $G_2$ which preserves the cross product and the inner product, it is clear that $S$ is a $\nabla$-parallel tensor, see [1, Lemma 2.4] for a direct proof. Note that $2A = SJ = -JS$ since $0 = D(J^2) = SJ + JS$.

Further, $S_v w = -S_w v$ by (7).

The torsion tensor of $\nabla$ is

\begin{equation}
T(v, w) = \nabla_v w - \nabla_w v - [v, w] = A_v w - A_w v .
\end{equation}

We have $2A_v w = S_v Jw = -JS_v w$, and thus $A_v w = -A_w v$. Hence

\begin{equation}
T(v, w) = S_v Jw = -JS_v w ,
\end{equation}

which shows again that $T$ is $\nabla$-parallel since so are $S$ and $J$.

We want to compute $S$ in terms our frame $(F, \bar{F})$. By (17), (13), and (15), $F_j$ is a real multiple of

\begin{equation}
F_j^o = (e_j - i fe_j) / \sqrt{2}
\end{equation}

where $e_1, e_2, e_3 \in \mathcal{O}'$ is an orthonormal 3-frame perpendicular to $f$ with $e_3 = e_1 e_2$. Since

\begin{align*}
(e_i - i fe_i)(e_j - i fe_j) &= 2(e_k + i fe_k) , \\
(e_i - i fe_i)(e_j + i fe_j) &= 0 , \\
(e_i - i fe_i)(e_i + i fe_i) &= -2 + 2i f .
\end{align*}

for $(i, j, k) = (1, 2, 3)$ up to cyclic permutations, we have from (25)

\begin{equation}
S_{F_i} F_j^o = \sqrt{2} \bar{F}_i^o , \quad S_{F_i} \bar{F}_j^o = 0 , \quad S_{\bar{F}_i} F_j^o = 0 .
\end{equation}

The real factors are given by (17). Thus

\textbf{Lemma 9.1.}

\begin{align*}
S_{F_1} F_2 &= \bar{F}_3 , \\
S_{F_2} F_3 &= 2 \mu \bar{F}_1 , \\
S_{F_3} F_1 &= 2 \lambda \bar{F}_2 , \\
S_{\bar{F}_j} F_k &= 0 \quad \forall j, k .
\end{align*}
Recalling $2A = SJ$ and $JF_j = iF_j$, we obtain:

**Corollary 9.1.** For $A' = A_{F_1}$ and $A'' = A_{F_1}$ we have

\[
\begin{align*}
2A'F_1 &= 0, \\
2A'F_2 &= iF_3, \\
2A'F_3 &= -2i\bar{F}_2, \\
2A''F_1 &= 0, \\
2A''F_2 &= 0, \\
2A''F_3 &= 0.
\end{align*}
\]

Next we compute the curvature tensor $R$ of $\nabla$, see also [10, Cor. 3.4]. From $\nabla v = Dv + Av$ we obtain when $[v, w] = 0$:

\[
R_{vw} = [\nabla v, \nabla w] = [Dv, Dw] + DvAw - DwAv + [Av, Aw].
\]

Here $[Dv, Dw] = R^o$ is the curvature tensor of the sphere $S^6$,

\[
(29) \quad R^o_{vw}x = \langle x, w \rangle v - \langle x, v \rangle w.
\]

Now $2A_w = (D_wJ)J = S_wJ$, hence $2DvAw = Dv(S_wJ) = (DvD_wJ)J + S_wS_v$. Thus

\[
2(DvAw - DwAv) = [Dv, Dw]J + [S_w, S_v],
\]

and moreover

\[
4[A_v, A_w] = [S_vJ, S_wJ] = [S_v, S_w]
\]

since $S_vJS_wJ = -S_vJJS_w = S_wS_v$. Thus

\[
(30) \quad R_{vw} = R^o_{vw} + (1/2)[R^o_{vw}, J] - (1/4)[S_v, S_w].
\]

Since $R^o$ is determined by the metric which is parallel and since $J$ and $S$ are parallel, we see directly that $R$ is parallel.

**Lemma 9.2.** For $R_{11} := R_{F_1 F_1} = [\nabla F_1, \nabla \bar{F}_1] = [\nabla', \nabla'']$ we have

\[
(31) \quad R_{11}F_1 = \lambda F_1, \quad R_{11}F_2 = -\frac{\lambda}{2} F_2, \quad R_{11}F_3 = -\frac{\lambda}{2} F_3
\]

\[R_{11}\bar{F}_1 = -\lambda \bar{F}_1, \quad R_{11}\bar{F}_2 = \frac{\lambda}{2} \bar{F}_2, \quad R_{11}\bar{F}_3 = \frac{\lambda}{2} \bar{F}_3.
\]

**Proof.** The first line follows from (30) with (29) and (28) where we put $v = F_1$ and $w = \bar{F}_1$. Applying $R^o_{11} = R^o_{F_1 F_1}$ to $F_1, F_2, F_3$ we observe $\langle F_1, F_j \rangle = 0$ and $\langle F_1, \bar{F}_j \rangle = \lambda \delta_{1j}$, hence

\[
(32) \quad R^o_{11}F_1 = \lambda F_1 \quad \text{while} \quad R_{11}F_2 = 0, \quad R_{11}F_3 = 0.
\]
In particular, \( R_{1\bar{1}} \) commutes with \( J \), and consequently the second term on the right hand side of (30) vanishes, \([ R_{1\bar{1}}, J ] = 0 \). It remains to compute \([ S_{F_1}, S_{\bar{F}_1}] \):

\[
S_{F_1} : F_2 \mapsto \bar{F}_3, \quad F_3 \mapsto -2\lambda \bar{F}_2,
\]
\[
S_{\bar{F}_1} : \bar{F}_2 \mapsto F_3, \quad \bar{F}_3 \mapsto -2\lambda F_2,
\]

while \( F_1, \bar{F}_1 \) are mapped to 0. Thus \([ S_{F_1}, S_{\bar{F}_1}] \) has eigenvalues \(-2\lambda\) for \( F_2, \bar{F}_3 \) and \( 2\lambda \) for \( \bar{F}_2, F_3 \) while \( F_1, \bar{F}_1 \) are mapped to 0. Now the first line of (31) follows from (30).

For the second line we just observe that \( R_{\bar{1}1} = -R_{1\bar{1}} \) and therefore \( R_{1\bar{1}} \bar{F}_j = R_{\bar{1}1} F_j = -R_{1\bar{1}} F_j \).

10. Structure equations

From Proposition 7.1 and Corollary 9.1 we obtain the derivatives of the Frenet frame:

**Proposition 10.1.** Let \( M \) be a Riemann surface and \( f : M \to \mathbb{S}^6 \) a pseudoholomorphic curve. Let \( \nabla \) denote the canonical \( G_2 \)-connection on \( \mathbb{S}^6 \) and let \( \nabla' = \nabla_{\partial/\partial z} \) and \( \nabla'' = \nabla_{\partial/\partial \bar{z}} \). Let \( F_1 = f_z, \ F_2 = f_{\bar{z}}, \ F_3 = f_{\bar{z}} \times f_{z\bar{z}} \) be the Frenet frame of \( f \). Then

\[
\nabla' F_1 = l_z F_1 + F_2,
\]
\[
\nabla' F_2 = m_z F_2 + \frac{ih}{2\lambda\mu} F_3,
\]
\[
\nabla' F_3 = 0,
\]
\[
\nabla'' F_1 = 0,
\]
\[
\nabla'' F_2 = -\frac{\mu}{\lambda} F_1,
\]
\[
\nabla'' F_3 = (i\hbar/\mu) F_2 + (l + m)_{\bar{z}} F_3.
\]

**Corollary 10.1.** The frame \( F = (f_z, f_{\bar{z}}, f_{\bar{z}} \times f_{z\bar{z}}) \) of \( E_+ = \{ v \in \mathbb{O}_c : f \times v = iv \} \) solves the differential equations

\[
\nabla' F = FB', \quad \nabla'' F = FB''
\]

with

\[
B' = \begin{pmatrix} l_z & 0 & 0 \\ 1 & m_z & 0 \\ 0 & \frac{ih}{2\lambda\mu} & 0 \end{pmatrix}, \quad B'' = \begin{pmatrix} 0 & -\frac{\mu}{\lambda} & 0 \\ 0 & 0 & i\hbar/\mu \\ 0 & 0 & (l + m)_{\bar{z}} \end{pmatrix}.
\]

**Remark 1.** In the superminimal case \( h = 0 \) we see that \( \nabla F_3 \) is a multiple of \( F_3 \). In our analogy with the Frenet frame of a space curve \( c \), the third vector \( F_3 \) corresponds to the binormal \( f_3 = f_1 \times f_2 \), where \( f_1 = c' \) and \( f_2 = (c')^\perp \), and \( f_3 \) is proportional to \( f_3 \) if and only if the torsion of \( c \) vanished (which means that \( c \) is a planar curve). Thus Bryant [3] calls superminimal pseudoholomorphic curves torsion free.
However, they are not “planar” in any sense: a weak analogue of planes would be a pseudoholomorphic embedding of a complex 2-dimensional manifold into \( S^6 \), but there are none. This makes these mappings particularly interesting.

Remark 2. One might wonder why the matrices \( B', B'' \) obviously do not belong to \( \mathfrak{su}_3 \) (though this was suggested in section 2). The reason is that the frame \( F \) is not normalized. This can easily be corrected by passing to the normalized frame \( F_0 \) with

\[
F_0 = F_0 D
\]

where

\[
D = \operatorname{diag}(\sqrt{\lambda}, \sqrt{\mu}, \sqrt{2\lambda\mu}).
\]

We have \( \nabla F = \nabla (F_0 D) = (\nabla F_0)D + F_0 \partial D \) and \( FB = F_0 DB \). Thus from \( \nabla F = FB \) we obtain \( \nabla F_0 = F_0 B_0 \) with

\[
B_0 = DBD^{-1} - (\partial D)D^{-1}.
\]

We have

\[
DB'D^{-1} = \begin{pmatrix}
\sqrt{\lambda} & 0 & 0 \\
0 & \sqrt{\mu} & 0 \\
0 & 0 & \sqrt{2\lambda\mu}
\end{pmatrix}
\begin{pmatrix}
l_x & 0 & 0 \\
1 & m_z & 0 \\
0 & i\frac{\hbar}{\sqrt{2\lambda\mu}} & 0
\end{pmatrix}
\begin{pmatrix}
l_x & 0 & 0 \\
1 & m_z & 0 \\
0 & i\frac{\hbar}{\sqrt{2\lambda\mu}} & 0
\end{pmatrix}
= \begin{pmatrix}
l_x & \frac{\hbar}{\sqrt{2\lambda\mu}} \\
\frac{\sqrt{\mu}}{\sqrt{\lambda}} & m_z \\
\frac{i\hbar}{\sqrt{2\lambda\mu}} & 0
\end{pmatrix},
\]

\[
(\partial' D)D^{-1} = \frac{1}{2} \operatorname{diag}(l_x, m_z, l_x + m_z), \quad \text{hence by (36),}
\]

\[
(37) \quad B'_o = \begin{pmatrix}
\frac{1}{2}l_x & 0 & 0 \\
\frac{\sqrt{\mu}}{\sqrt{\lambda}} & \frac{1}{2}m_z & 0 \\
0 & \frac{i\hbar}{\mu\sqrt{2\lambda}} & -\frac{1}{2}(l_x + m_x)
\end{pmatrix}.
\]

Similarly,

\[
(38) \quad B''_o = \begin{pmatrix}
-\frac{1}{2}l_x & -\frac{\sqrt{\mu}}{\sqrt{\lambda}} & 0 \\
0 & -\frac{1}{2}m_z & \frac{i\hbar}{\mu\sqrt{2\lambda}} \\
0 & 0 & \frac{1}{2}(l_x + m_x)
\end{pmatrix} = -(B'_o)^*.
\]

Recall that \( \nabla x = \nabla' + \nabla'' \) and \( \nabla y = i(\nabla' - \nabla'') \) where \( z = x + iy \) is the conformal coordinate. Thus

\[
(39) \quad \nabla x F^o = F^o (B'_o + B''_o), \quad \nabla y F^o = iF^o (B'_o - B''_o),
\]

and the matrices \( B'_o + B''_o \) and \( i(B'_o - B''_o) \) belong to \( \mathfrak{su}_3 \).

11. Integrability conditions

The coefficients of \( B' \) and \( B'' \) still must satisfy some relations, the integrability conditions for the overdetermined system (34). In fact,

\[
\nabla' \nabla'' F = \nabla'(FB'') = FB'' + FB''',
\]

\[
\nabla'' \nabla F = \nabla''(FB) = FB' + FB''
\]
\[ \nabla'' \nabla' F = \nabla'' (FB') = FB'' B' + FB'_z, \]

which implies
\[ [\nabla', \nabla''] F = F ([B', B''] + B'_z - B'') \]

On the other hand we have seen in Lemma 9.2:
\[ [\nabla', \nabla''] F = R_{11} F = F \text{diag}(\lambda, -\frac{\lambda}{2}, -\frac{\lambda}{2}). \]

Thus an integrability condition for (34) is
\[ (40) \text{diag}(\lambda, -\frac{\lambda}{2}, -\frac{\lambda}{2}) = R_{11} = [B', B''] + (B'')_z - (B')_z. \]

The commutator \([B', B'']\) equals
\[
\begin{pmatrix}
0 & 0 & 0 \\
1 & m_z & 0 \\
0 & \frac{ih}{2\lambda \mu} & 0
\end{pmatrix}
\begin{pmatrix}
0 & -\frac{\mu}{\lambda} & 0 \\
0 & 0 & i\frac{h}{\mu} \\
0 & 0 & (l + m)_{zz}
\end{pmatrix}
\begin{pmatrix}
l_z \\
m_z \\
\frac{ih}{2\lambda \mu}
\end{pmatrix},
\]
and the derivatives are
\[ (B'')_z = \begin{pmatrix}
0 & -\left(\frac{1}{\mu}\right)_z \\
0 & 0 \\
0 & 0
\end{pmatrix}, \quad (B')_z = \begin{pmatrix}
l_z & 0 & 0 \\
0 & m_z & 0 \\
0 & \frac{ih}{2} & 0
\end{pmatrix}.
\]

Since
\[ \left(\frac{1}{\mu}\right)_z = -\frac{m_z}{\mu}, \quad \left(\frac{1}{\lambda \mu}\right)_z = -\left(\frac{l + m}{\lambda \mu}\right) z, \quad \left(\frac{\mu}{\lambda}\right)_z = (m - l)_z \frac{\mu}{\lambda}, \]
we obtain from (40):
\[ (41) \quad \text{diag} \left(\lambda, -\frac{\lambda}{2}, -\frac{\lambda}{2}\right) = \text{diag} \left(\frac{\mu}{\lambda} - l_{zz}, \frac{|h|^2}{2\lambda \mu^2} - \frac{\mu}{\lambda} - m_{zz}, (l + m)_{zz} - \frac{|h|^2}{2\lambda \mu^2}\right). \]

**Lemma 11.1.** Let \(\lambda, \mu\) be absolute value type functions on \(M\) such that
\[ (42) \quad \mu = \lambda (l + z_{zz}) \]
and let \(h : M \to \mathbb{C}\) be a holomorphic function. Then (41) is satisfied if and only if
\[ (43) \quad |h|^2 = \lambda^2 \mu^2 + 2\lambda \mu^2 (l + m)_{zz}. \]
Proof. The condition (42) is equivalent to the equality in the first entry, and moreover, the equalities in the second and third entry become the same. The equality in the third entry is (43).

Lemma 11.2. If $F$ is the Frenet frame of a pseudoholomorphic curve $f : M \to S^6$ with Gaussian curvature $K$ and $h = \langle f_{zzz}, f_{zzz} \rangle$, then (43) is equivalent to

$$|h|^2 = \lambda^6(1 - K)^2(\Delta \log(1 - K) + 1 - 6K)$$

where $\Delta$ is the Laplacian of the induced metric on $M$.

Proof. We have

$$l + m = \log(\lambda \mu) \overset{(18)}{=} \log(\lambda^3(1 - K)) = 3\log \lambda + \log(1 - K).$$

Further, from $(\log \lambda)_{z\bar{z}} = -\lambda K$ and $\mu = \lambda^2(1 - K)$ (cf. (18)) and $\partial_z\partial_{\bar{z}} = \frac{1}{2}\lambda\Delta$ we obtain

$$2(l + m)_{z\bar{z}} = -6\lambda K + \lambda \Delta \log(1 - K)$$
$$2\lambda^2 \mu^2(l + m)_{z\bar{z}} = \lambda^2 \mu^2(-6K + \Delta \log(1 - K))$$
$$\lambda^2 \mu^2 + 2\lambda^2 \mu^2(l + m)_{z\bar{z}} = \lambda^2 \mu^2(1 - 6K + \Delta \log(1 - K))$$
$$= \lambda^6(1 - K)^2(1 - 6K + \Delta \log(1 - K)).$$

Thus the conditions (43) and (44) are the same.

12. Existence of pseudoholomorphic curves

Let $M \subset \mathbb{C}$ be an open domain. Suppose that on $M$ a holomorphic function $h$ and absolute value type functions $\lambda, \mu$ are given satisfying (18) and (43),

$$\mu = \lambda(\lambda + l_{z\bar{z}}),$$
$$|h|^2 = \lambda^2 \mu^2 + 2\lambda \mu^2(l + m)_{z\bar{z}},$$

where $l = \log \lambda$ and $m = \log \mu$. Over $M$ we consider the trivial vector bundle $E = M \times \mathbb{O}_c$ with a connection $\nabla$ defined by

$$\nabla' F = F B'$$
$$\nabla'' F = F B''$$

where $B', B''$ are given in (35),

$$B' = \begin{pmatrix} l_z & 0 & 0 \\ 1 & m_z & 0 \\ 0 & \frac{i\lambda}{2\mu} & 0 \end{pmatrix}, \quad B'' = \begin{pmatrix} 0 & -l_{z\bar{z}} & 0 \\ 0 & 0 & \frac{i\lambda}{\mu} \\ 0 & 0 & (l + m)_{z\bar{z}} \end{pmatrix},$$

and where $F = (F_1, F_2, F_3)$ with

$$F_1 = \sqrt{\lambda} F_1^o, \quad F_2 = \sqrt{\mu} F_2^o, \quad F_3 = \sqrt{2\lambda \mu} F_3^o,$$
$$F_1^o = (i - il)/\sqrt{2}, \quad F_2^o = (j - il)/\sqrt{2}, \quad F_3^o = (k - il)/\sqrt{2}.$$
In particular, the only nonzero derivatives are

\begin{align*}
\nabla' F_1 &= l z F_1 + F_2, \\
\nabla' F_2 &= m z F_2 + \frac{i h}{2\mu} F_3, \\
\nabla'' F_2 &= -(\lambda + l z) F_1, \\
\nabla'' F_3 &= \frac{i h}{\mu} F_2 + (l + m) z F_3.
\end{align*}

On $E$ we have the tensor fields $J, S, T, R$ where

\[ Jv = l \times v, \quad S_v w = (v \times w)^T \]

with $T = \text{Span}(i, j, k, li, lj, lk)$, and where $T, R$ are given by (26), (30), (29). In order to apply the existence and uniqueness theorem in [6] we need that $\nabla$ is a metric connection and $J, S$ (and hence $T, R$) are parallel with respect to $\nabla$. This follows by passing to the normalized frame $F^o$ and using that $B^o + B''^o$ and $i(B^o - B''^o)$ belong to the Lie algebra $\mathfrak{su}_3$ acting on $\mathbb{C}^3 = T$ with $l$ as complex structure, see (37), (38) (Remark 2 in section 10). The holonomy group $SU_3$ preserves the metric and the tensors $J$ and $S$, hence $R$.

We are ready now to reprove Hashimoto’s result [9].

**Theorem 12.1.** Let $M$ be a simply connected Riemann surface carrying a compatible Riemannian metric $ds^2$, possibly with branch points, and a holomorphic 6-form $\Lambda$. Let $K$ be the Gaussian curvature and $\Delta$ the Laplacian of $ds^2$. Suppose that $1 - K$ is an absolute value type function.

Then there is a unique pseudoholomorphic curve $f : M \to S^6$ (up to translation with elements of $G_2$) such that $ds^2$ is the induced metric and $\Lambda = \Lambda_3$ is the third Hopf differential (see section 6) if and only if

\[ (1 - K)^2(\Delta \log(1 - K) + 1 - 6K) = |\Lambda|^2. \]

**Proof.** $\Rightarrow$ If such a pseudoholomorphic curve $f : M \to S^6$ is given, then (46) is satisfied by Lemma 11.1 and (44), note that

\[ |\Lambda|^2 = |h|^2 / \lambda^6. \]

$\Leftarrow$: Let $(M, ds^2)$ and $\Lambda$ be given with (46). Choosing a conformal coordinate $z$ on some simply connected open subset $M_o \subset M$, we have $ds^2 = 2\lambda dz d\bar{z}$ for some absolute value type function $\lambda$, and $\Lambda = h(z)dz^6$

\[ A \] compatible Riemannian metric of a Riemann surface is locally of the type $dz^2 = 2\lambda dz d\bar{z}$ for some conformal coordinate $z$ on $M$, where $\lambda$ is a positive function. If we allow for isolated zeros of $\lambda$ such that $\lambda$ is an absolute value type function, such zeros are called branch points of the metric $ds^2$.\[ A \]
for some holomorphic function $h$ with (47). Further we put $l = \log \lambda$ and define the absolute value type function

$$\mu = \lambda (\lambda + l \overline{z})^{18} = \lambda^2 (1 - K).$$

Using these functions, we consider the bundle $E = M_o \times T$ with $T = \text{Span}_C(i, j, k, il, jl, kl)$ with sections $F_1, F_2, F_3$ and a connection $\nabla$ as defined in (45) at the beginning of this section. By the main theorem of [6], there exists a smooth map $f : M \to S^6$ and a bundle isomorphism $\Phi : E \to f^*TS$ preserving the metric and the tensors $J, S, R$ such that

$$\Phi \circ f \overline{z} = F_1$$

if and only if

$$\nabla' F_1 - \nabla'' F_1 = T(F_1, \overline{F}_1) = 0,$$

$$[\nabla', \nabla''] F = R_{F_1 F_1} F = F \text{ diag}(\lambda, -\frac{\lambda}{2}, -\frac{\lambda}{2}).$$

The first equation holds by (45) since $\nabla' F_1 = 0 = \nabla'' F_1$.

The second equation comes down to (40) and (41) which in turn is equivalent to (44) or (46), by Lemma 11.1. This proves existence and uniqueness of a pair of maps $(f, \Phi)$ satisfying (48), and $f$ is pseudoholomorphic since $F_1$ and $f \overline{z}$ lie in the $i$-eigenspace of $J$. Moreover, $F = (F_1, F_2, F_3)$ becomes the Frenet frame along $f$ (via $\Phi$), using (45). In particular, from the “$\Rightarrow$”-part we see $h = \langle (F_2)_z, (F_2)\overline{z} \rangle$, cf. (19). This finishes the proof.

**Remark.** Replacing $\Lambda$ by $e^{i\theta} \Lambda$ for some constant angle $\theta$ does not change the condition (46). This gives the associated family of the minimal surface $f$ which also consists of pseudoholomorphic curves.

**Corollary 12.1.** Let $(M, ds^2)$ be as in the assumptions of the Theorem 12.1. Then there is a superminimal (“torsion free”) pseudoholomorphic curve $f : M \to S^6$, unique up to translations in $G_2$, with induced metric $ds^2$ if and only if

$$\Delta \log (1 - K) = 6K - 1.$$

13. Pseudoholomorphic curves in $S^5$

Another interesting special case is when a pseudoholomorphic curve $f : M \to S^6$ actually takes values in some equator sphere $S^5 \subset S^6$. We will call it a pseudoholomorphic curve in $S^5$.

**Lemma 13.1.** Let $f : M \to S^6$ be a pseudoholomorphic curve and $z$ a conformal coordinate on $M$. Then $f$ takes values in some great sphere $S^5 \subset S^6$ if and only if

$$|h| = \lambda \mu.$$
Proof. Assume that \( f \) lies in \( S^5 \). Then there exists a constant unit vector \( \xi \) (inside \( N_2 \)) such that \( \langle f, \xi \rangle = 0 \). Using \( f_z, f_{zz}, f_{zzz} \perp \xi \) and (21) we obtain
\[
\lambda \mu \langle f_z \times f_{zz}, \xi \rangle = h \langle f_z \times f_{zz}, \xi \rangle
\]
and by conjugation
\[
\lambda \mu \langle f_z \times f_{zz}, \xi \rangle = \bar{h} \langle f_z \times f_{zz}, \xi \rangle.
\]
Multiplying these two equations we find \( |h| = \lambda \mu \).

Conversely, we assume that \( |h| = \lambda \mu \). Then comparing (21) and its conjugate we obtain a linear relation between \( ((f_{zz})^\xi_z)^{N_2} \) and its conjugate:
\[
(52) \quad \frac{h}{\lambda \mu} ((f_{zz})^\xi_z)^{N_2} = ((f_{zz})^\xi_z)^{N_2}.
\]
Thus the real and the imaginary part of \( ((f_{zz})^\xi_z)^{N_2} \) are linearly dependent, and hence there is a real unit vector \( \xi \in N_2 \) which is perpendicular to \( ((f_{zz})^\xi_z)^{N_2} \). Consequently, \( \xi \) is perpendicular to all derivatives of \( f \) up to third order, and hence \( \xi_z \perp f, f_z, f_{zz}, f_{zzz}, \xi \). So \( \xi_z \) must be a multiple of \( ((f_{zz})^\xi_z)^{N_2} \), and by (52) the same holds for \( \xi_z \). On the other hand, \( \langle \xi_z, (f_{zz})^\xi_z \rangle = -\langle \xi, (f_{zz})_z \rangle = 0 \) since from \( f_{zz} = f_{zzz} + l_z f_z \) we obtain \( (f_{zz})_{zz} = f_{zzzz} + (l_z f_z)_{zz} \in \text{Span}(f, f_z, f_{zz}) \perp \xi_z \). Thus \( \xi \) is a constant vector and we conclude that \( f \) lies in \( S^5 = S^6 \cap \xi^\perp \).

Theorem 13.1. Let \( M \) be a simply connected Riemann surface with compatible metric \( ds^2 \) (possibly with branch points), and let \( K \) be its Gaussian curvature and \( \Delta \) its Laplacian. Suppose that \( 1 - K \) is an absolute value type function. Then there is an isometric pseudoholomorphic map \( f : M \to S^5 \) if and only if
\[
(53) \quad \Delta \log(1 - K) = 6K.
\]
In fact, up to translations with elements of \( G_2 \) there is precisely one associated family of such maps.

Proof. If \( f : M \to S^5 \) is pseudoholomorphic with induced metric \( ds^2 = 2 \lambda dz \bar{d}z \), we have \( |h| = \lambda \mu \) and \( |h|^2 = \lambda^2 \mu^2 = \lambda^6(1 - K)^2 \) using \( \mu = \lambda^2(1 - K) \). Thus the integrability condition
\[
(44) \quad \lambda^6(1 - K)^2(\Delta \log(1 - K) + 1 - 6K) = |h|^2
\]
becomes (53). Conversely, (53) becomes (44) when we put \( |h| := \mu \lambda = \lambda^3(1 - K) \). Then
\[
\Delta \log |h| = 3 \Delta \log \lambda + \Delta \log(1 - K) = 0,
\]
using (53) and the relation between conformal factor and curvature, 
\[ \Delta \log \lambda = -2K. \]
Thus \( \log |h| \) is harmonic, hence the real part of a holomorphic function, and \( |h| \) is the absolute value of a holomorphic function \( h \), uniquely determined up to some constant phase factor \( e^{i\theta} \).
Thus \( \Lambda = hdz^6 \) defines a holomorphic 6-form, and we conclude from Theorem 12.1 that there is a pseudoholomorphic map \( f : M \to S^6 \) with induced metric \( ds^2 \). Since \( |h| = \lambda \mu \), we see from Lemma 13.1 that \( f \) takes values in some great sphere \( S^5 \subset S^6 \).

\[ \square \]

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