THE SPECTRAL PARAMETER OF PLURIHARMONIC MAPS

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Abstract. An important tool for studying pluriharmonic maps with values in compact symmetric spaces is the spectral family, which comes in several versions: extended solutions, extended frames and associated families. In this paper we describe the relations between these notions.

1. Introduction

What is an integrable system? Although this notion seems a bit vague, one of the common features is that the differential equation allows for a one-parameter deformation, depending on the so-called spectral parameter λ. This is often introduced in a purely formal way. It is the purpose of the present article to discuss the geometric meaning of λ in an important case, that of harmonic maps of surfaces and pluriharmonic maps of complex manifolds with values in Riemannian symmetric spaces. We shall link the spectral parameter to the associated family of such maps, which is well known from elementary minimal surface theory; the most prominent example is the deformation of the catenoid into the helicoid. In fact, our geometric theory joins two different approaches to (pluri-) harmonic maps: extended solutions and extended frames.

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2. Extended solutions and extended frames

The equation of a harmonic map f of a Riemann surface M into a compact (not necessarily connected) Lie group G with biinvariant metric or a totally geodesic submanifold $S \subset G$ (a symmetric space) allows for a spectral parameter $\lambda \in \mathbb{S}^1$. There are two different ways to assign to f a λ-dependent map, a so called spectral deformation.

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The first one goes back to Uhlenbeck [U], motivated by earlier work in physics [P, ZM, ZS], the second one was introduced by Burstall and Pedit [BP], see also [DPW].

Uhlenbeck introduced the notion of an *extended solution*. This is a family of maps $\Phi_\lambda : M \to G$ depending smoothly on $\lambda \in \mathbb{S}^1$ such that Φ_1 is a constant group element,[1] and the Maurer-Cartan form[2]

$$\beta_\lambda := \Phi_\lambda^{-1} d\Phi_\lambda \in \Omega^1(M, \mathfrak{g}) \tag{1}$$

satisfies

$$\beta_\lambda = (1 - \lambda^{-1})\beta' + (1 - \lambda)\beta'' \tag{2}$$

for some $\beta' \in \Omega^{(1,0)}(M, \mathfrak{g}^c)$ and $\beta'' = \overline{\beta'}$.[3] By [U], a map $f : M \to G$ is harmonic iff there exists (at least locally) an extended solution Φ_λ with $f = \Phi_{-1}$. Since the inversion $j : G \to G$, $j(g) = g^{-1}$ is an isometry of G for any biinvariant metric, $f^{-1} = j \circ f$ is again harmonic, and, up to left translation, the corresponding extended solution is (cf. [BG])

$$(T\Phi)_\lambda := \Phi_{-\lambda}\Phi_{-1}^{-1}. \tag{3}$$

The map f may take values in a totally geodesic submanifold $S \subset G$ and can then be considered as a harmonic map into S rather than into G. In particular we consider a *Cartan embedded symmetric space* S which is a connected component of the set of order 2 elements,

$$\sqrt{e} = \{s \in G;\ s^2 = e\} \subset G,$$

(*standard* Cartan embedding) or a left translate of such a set.[4]

The approach by extended frames in turn uses the projection $\pi : G \to S = G/K$ rather than a Cartan embedding $\iota : S \to G$. Starting with a map $f : M \to S$, one first chooses a lift ("frame") $F : M \to G$ with $f = \pi \circ F$; in fact F may be defined only locally, on some open subset $M_o \subset M$. If $S \subset G$ is standard Cartan-embedded, then G acts by conjugation, and the relation between F and f is given by

$$f = F s_o F^{-1} \tag{4}$$

[1] Often one assumes $\Phi_{-1} = e$ (unit element in G) but we would like to make the notion independent under left translations in G.

[2] Using matrix notation, we write gx for the left translation $L_g x$ (for $x \in G$) as well as for its differential $dL_g x$ (for $x \in \mathfrak{g}$).

[3] By $\Omega^{(1,0)}$ we denote the space of one-forms ω which are complex linear, $\omega(JX) = i\omega(X)$ for any $X \in TM$, while its complex conjugate $\bar\omega \in \Omega^{(0,1)}$ is anti-linear, $\bar\omega(JX) = -i\bar\omega(X)$.

[4] A standard Cartan embedding $(\sqrt{e})_o$, i.e. a connected component of $\sqrt{e}$, may or may not be contained in the identity component G_o of G. Using a left translation in G it can be shifted into G_o. If one uses the left translation by s_o for some $s_o \in (\sqrt{e})_o$, then $s_o(\sqrt{e})_o = \{s_o s;\ s \in (\sqrt{e})_o\}$ is called *pointed Cartan embedding.*

for some $s_o \in \sqrt{e}$ (which is the point reflection at $o = eK \in S$).

Let $\alpha = F^{-1}dF \in \Omega^1(M_o, \mathfrak{g})$ be the corresponding Maurer-Cartan form. We decompose $\alpha = \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{p}}$ according to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ corresponding to $\mathrm{Ad}(s_o)$, i.e. $\mathrm{Ad}(s_o) = I$ on $\mathfrak{k}$ and $\mathrm{Ad}(s_o) = -I$ on $\mathfrak{p}$. Then f is harmonic iff the modified one-form

$$\alpha_\lambda = \alpha_{\mathfrak{k}} + \lambda^{-1}\alpha'_{\mathfrak{p}} + \lambda\alpha''_p \tag{5}$$

is integrable, i.e.

$$\alpha_\lambda = F_\lambda^{-1}dF_\lambda \tag{6}$$

for some smooth map $F_\lambda : M_o \to G$ depending smoothly on $\lambda \in \mathbb{S}^1$ with $F_1 = F$; this is called an *extended frame*. Moreover, all maps $f_\lambda := \pi \circ F_\lambda : M_o \to S$ are harmonic.

Both approaches have been extended from harmonic maps of surfaces to pluriharmonic maps of Kähler manifolds, cf. [OV] for Uhlenbeck's theory and [DE] for the extended frame method. A map f on a Kähler manifold[5] M is called *pluriharmonic* if its restriction to any complex curve in M is harmonic, or, in more technical terms, if its Levi form $Ddf^{(1,1)}$ (the (1,1) part of its hessian) vanishes. Everything we have said remains unchanged after replacing the word "harmonic" everywhere by "pluriharmonic".

The two approaches are related to each other by the following

Theorem 1. *Let $f : M \to S \subset G$ be a pluriharmonic map where $S = G/K$ is (standard) Cartan embedded into G. Let $F : M_o \to G$ be a local frame for f, i.e. $f = \pi \circ F = F s_o F^{-1}$. Then extended solutions and extended frames for f are related by*

$$F_\lambda = \Phi_\lambda F. \tag{7}$$

More precisely, suppose that two families of maps $F_\lambda, \Phi_\lambda : M_o \to G$, $\lambda \in \mathbb{S}^1$, satisfying (7), are given. Then F_λ is a an extended frame for f if and only if Φ_λ is an extended solution for f.

Proof. Differentiation of (7) yields

$$\alpha_\lambda = F^{-1}\beta_\lambda F + \alpha \tag{8}$$

where $\alpha_\lambda = F_\lambda^{-1}dF_\lambda$, $\alpha = \alpha_1 = F^{-1}dF$ and $\beta_\lambda = \Phi_\lambda^{-1}d\Phi_\lambda$. Using (4) we relate f to α:

$$f^{-1}df = F(s_o\alpha s_o - \alpha)F^{-1} = -2\tilde{\alpha} \tag{9}$$

[5]In fact it suffices that M is a complex manifold; locally we may always choose a Kähler metric and the notion of pluriharmonicity is independent of the choice of that metric.

where

$$\tilde\alpha := F\alpha_{\mathfrak p}F^{-1}; \tag{10}$$

remind that $s_o(\alpha_{\mathfrak k}+\alpha_{\mathfrak p})s_o = \alpha_{\mathfrak k}-\alpha_{\mathfrak p}$.

Now suppose that F_λ is an extended frame for f. Then from (5) and (8) we obtain

$$\beta_\lambda = F(\alpha_\lambda-\alpha)F^{-1} = (\lambda^{-1}-1)\tilde\alpha' + (\lambda-1)\tilde\alpha''. \tag{11}$$

Thus $\beta_\lambda = \Phi_\lambda^{-1}d\Phi_\lambda$ satisfies (2) with $\beta = -\tilde\alpha$. In particular, $\beta_1 = 0$, hence $\Phi_1 = const$, and

$$\Phi_{-1}^{-1}d\Phi_{-1} = \beta_{-1} = -2\tilde\alpha \overset{(9)}{=} f^{-1}df \tag{12}$$

whence $\Phi_{-1} = f$ up to a left translation. Thus Φ_λ is an extended solution for f.

Vice versa, suppose that Φ_λ is an extended solution for $f=\Phi_{-1}$. Let $F_\lambda = \Phi_\lambda F$ and $\alpha_\lambda = F_\lambda^{-1}dF_\lambda$. From (8) and (2) we obtain

$$\alpha_\lambda - \alpha = F^{-1}\beta_\lambda F = (1-\lambda^{-1})\tilde\beta' + (1-\lambda)\tilde\beta'' \tag{13}$$

where

$$\tilde\beta_\lambda = F^{-1}\beta_\lambda F. \tag{14}$$

We show first that the right hand side of (13) takes values in (the complexification of) $\mathfrak p$. In fact, from (9) we have on the one hand

$$F^{-1}(f^{-1}df)F = -2\alpha_{\mathfrak p}, \tag{15}$$

and on the other hand, by (2) for $\lambda=-1$ and (14),

$$F^{-1}(f^{-1}df)F = F^{-1}(\Phi_{-1}^{-1}d\Phi_{-1})F = F^{-1}\beta_{-1}F = 2\tilde\beta. \tag{16}$$

Thus

$$\tilde\beta = -\alpha_{\mathfrak p} \in \mathfrak p, \tag{17}$$

and hence $\alpha_\lambda - \alpha$ takes values in $\mathfrak p$, cf. (13). This implies that the $\mathfrak k$-components of α_λ and α are the equal. But (13) shows more:

$$\begin{aligned}(\alpha_\lambda)_{\mathfrak p} &= -\tilde\beta + (1-\lambda^{-1})\tilde\beta' + (1-\lambda)\tilde\beta'' \\ &= -\lambda^{-1}\tilde\beta' - \lambda\tilde\beta'' \\ &= \lambda^{-1}\alpha_{\mathfrak p}' + \lambda\alpha_{\mathfrak p}'',\end{aligned}$$

hence we have proved (5). $\square$

3. Associated families

We want to outline a third approach [ET2] which gives a geometric interpretation of both extended solutions and extended frames in a single theory. The starting point is the observation that a pluriharmonic map f allows a one-parameter deformation of pluriharmonic maps f_λ, the so called *associated family*.

It has already been observed by Weierstraß that minimal surfaces in euclidean space come in one-parameter families, so called *associated families*. The best known example is the deformation of the catenoid into the helicoid where we cut the catenoid along a vertical meridian and then move the two ends of the cut upwards and downwards apart from each other.[6] Starting with a surface f, the associated family is an isometric deformation f_θ with three properties, shown in the pictures:

- Up to parallel translation, each tangent plane remains unchanged during the deformation,
- Principal curvatures are preserved while principal curvature lines rotate,
- The deformation is periodic, $f_{\theta+2\pi} = f_\theta$, and after a half period π we see the same object in opposite orientation.

In fact, denoting by R_θ the rotation by the angle θ in the tangent plane of the surface, the above properties are expressed by

$$df \circ \mathsf{R}_\theta = df_\theta. \tag{18}$$

One may ask which (other) surfaces $f : M \to \mathbb{R}^n$ allow an associated family (18). It is enough to consider the 90^o rotation $J = \mathsf{R}_{\pi/2}$ since

$$\mathsf{R}_\theta = (\cos\theta)I + (\sin\theta)J. \tag{19}$$

We need to find a map $g : M \to \mathbb{R}^n$ with

$$df \circ J = dg.$$

If M is simply connected (which we will always assume), this is equivalent to

$$d(df \circ J) = 0.$$

From $df = f_x dx + f_y dy$ we see $df \circ J = f_y dx - f_x dy$ and hence

$$d(df \circ J) = f_{yy} dx \wedge dy - f_{xx} dy \wedge dx = \Delta f \, dx \wedge dy$$

where $\Delta f = \text{trace } Ddf = f_{xx} + f_{yy}$ is the Laplacian. Hence an associated family exists iff f is *harmonic*, $\Delta f = 0$. In particular this applies to minimal surfaces which are just conformal harmonic maps.

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[6] http://page.mi.fu-berlin.de/polthier/Calendar/Kalender86/Kalender86.htm

Now we replace euclidean space $\mathbb{R}^n$ by an arbitrary symmetric space $S = G/K$. Furthermore we replace the surface by a Kähler manifold M, i.e. a Riemannian manifold with a parallel almost complex structure J. We can still define the parallel tensors R_θ by (19) and ask the following question: Given a smooth map $f : M \to S$, what is the condition that the one-form $df \circ \mathsf{R}_\theta$ is integrable, i.e. the differential of a map f_θ? To make this question more precise recall that df_x is a linear map from T_xM to $T_{f(x)}S$ and hence it defines a bundle map $df : TM \to f^*TS$. Thus (18) has to be modified since f_θ^*TS and f^*TS are different vector bundles. What we need is an isomorphism Φ_θ between these bundles such that

$$df_\theta = \Phi_\theta \circ df \circ \mathsf{R}_\theta, \tag{20}$$

where the isomorphism $\Phi_\theta : f^*TS \to f_\theta^*TS$, like parallel translation in euclidean $\mathbb{R}^n$, preserves the metric and the Lie triple structure (curvature tensor) on TS and is parallel with respect to the induced connections on f^*TS and f_θ^*TS. A family of pairs (f_θ, Φ_θ) with $f_0 = f$, $\Phi_0 = I$ satisfying (20) will be called an *associated family* for $f : M \to S$.

In [ET1], the question has been discussed in a more general setting: Let there be a vector bundle $E \to M$ and an E-valued one-form (bundle homomorphism) $\varphi : TM \to E$. Suppose that the fibres of E carry a connection ∇ and a parallel Lie triple structure R^S on the fibres which is isomorphic to the Lie triple structure of a Riemannian symmetric space S. When does there exist a smooth map $f : M \to S$ with $df = \varphi$, using a suitable isomorphism $\Phi : f^*TS \to E$? The answer was given in [ET1]: f exists and is unique up to isometries of S if and only if $d^\nabla \varphi = 0$ and $R^\nabla = \varphi^* R^S$. The proof is an application of the Frobenius integrability theorem. Applying this theory to $\varphi_\theta = df \circ \mathsf{R}_\theta$ one obtains

Theorem 2. [ET2] *Let M be a Kähler manifold, S a compact symmetric space and $f : M \to S$ a smooth map. There exists an associated family (f_θ, Φ_θ) for f (unique up to isometries of S) if and only if f is pluriharmonic.*

In this case, Φ_θ is an isometric bundle isomorphism between f^*TS and f_θ^*TS (i.e. it maps $T_{f(x)}S$ isometrically onto $T_{f_\theta(x)}S$ for any $x \in M$) which is parallel and preserves the curvature tensor R^S of S. Thus $\Phi_\theta(x)$ is the differential of an isometry[7] of S. Since an isometry is

[7]If S is not simply connected, this in not true in general, but it is true for the identity component of the isometry group; note that $\Phi_\theta(x)$ can be connected to $\Phi_0(x) = I$.

uniquely determined by its differential at a single point, Φ_θ may be viewed as an element of the isometry group G of S.

In [DE] the connection to the extended frames was given:

Theorem 3. [DE] *Let $f : M \to S = G/K$ be pluriharmonic with (local) frame $F : M_o \to G$ and associated family $(f_\lambda, \Phi_\lambda)$. Put $\lambda = e^{-i\theta}$. Then*

$$F_\lambda = \Phi_\lambda F \tag{21}$$

is an extended frame in the sense of [BP, DPW].

The idea of the proof is as follows. We have to show that $\alpha_\lambda = F_\lambda^{-1} dF_\lambda$ satisfies (5). We split $\alpha_\lambda = \alpha_{\mathfrak{k}}^\lambda + \alpha_{\mathfrak{p}}^\lambda$. The property $\alpha_{\mathfrak{k}}^\lambda = \alpha_{\mathfrak{k}}$ is obtained as follows. From (8) we have

$$\alpha_\lambda - \alpha = F^{-1}\beta_\lambda F \tag{22}$$

and the right hand side of (22) takes values in $\mathfrak{p}$, due to the parallelism of Φ_λ (see subsequent Lemma). Moreover, $d\pi(F_\lambda \alpha_{\mathfrak{p}}^\lambda) = df_\lambda$ where $\pi : G \to G/K$ is the projection, and $df_\lambda = \Phi_\lambda \circ df \circ \mathsf{R}_{-\theta}$. Since $T'M$ and $T''M$ are the $\pm i$ eigenspaces of J and the $e^{\mp i\theta}$ eigenspaces of $\mathsf{R}_{-\theta}$, we have $df \circ \mathsf{R}_{-\theta} = \lambda^{-1} d'f + \lambda\, d''f$ which shows (5). □

Lemma 1. *Let $f, \tilde{f} : M \to S = G/K$ be smooth maps and $\Phi : f^*TS \to \tilde{f}^*TS$ an isometric bundle isomorphism preserving R^S. Then Φ is parallel iff $F^{-1}(\Phi^{-1}d\Phi)F$ takes values in $\mathfrak{p}$ for any frame F of f.*

Proof. Let $x(t)$ be any smooth curve in M with $x(t_o) = x_o$ fixed. Consider the curves $c(t) = f(x(t))$ and $\tilde{c}(t) = \tilde{f}(c(t))$ in S. Then Φ is parallel iff $\Phi(t) := \Phi(x(t))$ maps parallel frames along c onto parallel frames along $\tilde{c}$. Parallel frames along a curve c in $S = G/K$ are the horizontal lifts C of c where horizontal subspaces in TG are left translates of $\mathfrak{p}$. In other words, $C(t) \in G$ with $\pi \circ C = c$ where $\pi : G \to G/K$, and $C(t)^{-1}C'(t) \in \mathfrak{p}$. Likewise,[8] $\tilde{C}(t) = \Phi(t)C(t)$ is horizontal iff $\tilde{C}(t)^{-1}\tilde{C}'(t) \in \mathfrak{p}$. But

$$\tilde{C}' = (\Phi C)' = \Phi' C + \Phi C'$$

[8]Again we consider $\Phi(t)$ as an element of G. A similar identification takes place for frames: Strictly speaking, a frame $C(t)$ at $p = f(x(t))$ is a basis of T_pS which arises by applying some $g \in G$ (more precisely: its differential dg_o) to a fixed basis $e_1, \dots, e_n$ of T_oS where $o = eK \in G/K$. Usually we identify $C(t)$ with g. The equality $\tilde{C}(t) = \Phi(t)C(t)$ can be understood in two ways, linked by the chain rule: either both $\tilde{C}(t)$ and $C(t)$ are considered as n-tuples of tangent vectors, mapped onto each other by the homomorphism $\Phi(t)$, or $\tilde{C}(t), \Phi(t), C(t) \in G$ and the right hand side is a product of group elements. We shall adopt the second view point.

and therefore

$$\tilde{C}^{-1}\tilde{C}' = (\Phi C)^{-1}(\Phi C)' = C^{-1}\Phi^{-1}\Phi' C + C^{-1}C'$$

The second term on the right hand side lies in $\mathfrak{p}$, due to horizontality of C. Thus $\tilde{C}$ is horizontal iff $C^{-1}\Phi^{-1}\Phi' C \in \mathfrak{p}$. Choosing $C(t_o) = F(x_o)$ we have proved our claim. □

From theorems 1 and 3 we obtain:

Theorem 4. *Let $S \subset G$ be a Cartan embedded symmetric space and $f : M \to S$ pluriharmonic with associated family $(f_\lambda, \Phi_\lambda)$. Then Φ_λ is an extended solution of f.*

Thus the theory of associated families $(f_\lambda, \Phi_\lambda)$ combines aspects of both theories: Φ_λ is the extended solution and $F_\lambda = \Phi_\lambda F$ the extended frame with $\pi \circ F_\lambda = f_\lambda$. Moreover we have achieved a geometric interpretation of Φ_λ which persists if no embedding of S into G is given: Φ_λ is the isomorphism between the bundles f^*TS and f_λ^*TS which one needs to define the associated family, cf. (20).

Note that a solution $(f_\lambda, \Phi_\lambda)$ of (20) for any single λ is unique up to left translation with some $g_\lambda \in G$. In particular, for $\lambda = -1$ or $\theta = \pi$ we have $\mathsf{R}_\pi = -I$ and thus ($f_{-1} = f$, $\Phi_{-1} = -I$) is a special solution with $\Phi_{-1}(x) = s_{f(x)} \in G$ (geodesic reflection at the point $f(x)$). Thus a general solution will be a left translate of this map, and hence we see immediately that Φ_{-1} is the composition of f with a Cartan embedding (which follows also from Theorem 3).

4. Totally geodesic submanifolds in Lie groups

So far, we have linked extended solutions, extended frames and associated families only when the symmetric space S is Cartan embedded in a Lie group G. But all three theories extend beyond this case. Extended solutions Φ_λ take values in a compact Lie group G and $f = \Phi_{-1}$ may lie in any closed totally geodesic submanifold $S \subset G$ (not only Cartan embeddings), while extended frames and associated families do not make use of embeddings of S at all. So let us assume that $S \subset G$ is a general closed totally geodesic submanifold and $f : M \to S \subset G$ a pluriharmonic map. Is there still a relation between extended solutions Φ_λ and extended frames F_λ of f? This seems unclear because Φ_λ and F_λ take values in different groups: While Φ_λ is G-valued, F_λ maps into the transvection group of S which now will be called H (rather than G).

But there is a link between the two groups: H is finitely covered by the group of the transvections of G keeping $S \subset G$ invariant. The group $G\times G$ acts on G by left and right translation, $(g_1,g_2)g = g_1 g g_2^{-1}$, and this action (after dividing out the ineffective diagonal of the center) is the transvection group of G. In fact, by definition a transvection on G is the composition of any two point reflections $s_g, s_{\tilde g}$ for $g,\tilde g\in G$. We have $s_g(p) = gp^{-1}g$ and hence

$$s_g(s_{\tilde g}(p)) = g(\tilde g p^{-1}\tilde g)^{-1} g = g\tilde g^{-1} p \tilde g^{-1} g \tag{23}$$

for any $p\in G$. Thus $s_g s_{\tilde g}$ is the action on G of $(g\tilde g^{-1}, g^{-1}\tilde g)\in G\times G$.

If $S\subset G$ is a closed totally geodesic submanifold, the transvections along S are just restrictions to S of the transvections $s_g s_{\tilde g}$ with $g,\tilde g\in S$. Thus the transvection group H of S is finitely covered by the group $\tilde H\subset G\times G$ generated by the set

$$\Gamma = \{(g\tilde g^{-1}, g^{-1}\tilde g);\ g,\tilde g\in S\}\subset G\times G. \tag{24}$$

The extended frames F_λ of a pluriharmonic map $f : M\to S$ take values in H. They will be lifted to $\tilde H$ and then called $\tilde F_\lambda$. Thus $\tilde F_\lambda$ and $\tilde\Phi_\lambda = \tilde F_\lambda \tilde F_1^{-1}$ take values in $\tilde H$. Since $\tilde\Phi_{-1}(x)$ acts on S as $s_o s_{f(x)}$ for a fixed $o\in S$, we may assume $\tilde\Phi_{-1}(x) = (of(x)^{-1}, o^{-1}f(x))$ for all $x\in M$.

On the other hand, we have the Uhlenbeck extended solution $\Phi_\lambda : M\to G$ with $\Phi_{-1} = f : M\to S\subset G$. Embedding S totally geodesically into $\tilde H\subset G\times G$ via[9]

$$i_o : S\ni g\mapsto (og^{-1}, o^{-1}g)\in\Gamma\subset\tilde H\subset G\times G, \tag{25}$$

we obtain a pluriharmonic map $i_o\circ f = (of^{-1}, o^{-1}f) : M\to G\times G$. This is a left translate in $G\times G$ of the pluriharmonic map (f^{-1}, f) with the extended solution

$$\hat\Phi_\lambda = ((T\Phi)_\lambda, \Phi_\lambda), \tag{26}$$

cf. (3). Thus we have two extended solutions $\tilde\Phi_\lambda, \hat\Phi_\lambda$ for (left translates of) $i_o\circ f$ which by unicity must agree up to left translations in $G\times G$. We have proved:

Theorem 5. *Let $S\subset G$ be totally geodesic submanifold and $f : M\to S$ a pluriharmonic map. Then G-valued extended solutions Φ_λ and H-valued extended frames F_λ for f correspond in the sense*

$$((T\Phi)_\lambda, \Phi_\lambda) = \tilde F_\lambda \tilde F_1^{-1} \tag{27}$$

up to left translations in $G\times G$, where $\tilde F_\lambda$ is a lift of F_λ to $\tilde H\subset G\times G$.

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[9] i_o is a lift to $\tilde H$ of the pointed Cartan embedding $\iota_o : S\to H$, $p\mapsto s_o s_p$.

Remark. If $S \subset G$ is standard Cartan embedded, i.e. $S = (\sqrt{e})_o$, then $\Gamma = \{(s\tilde{s}, s\tilde{s});\ s, \tilde{s} \in (\sqrt{e})_o\} \subset \Delta G \subset G \times G$, see (24). On the other hand an extended solution Φ_λ with $\Phi_{-1} \in \sqrt{e}$ can be chosen to be invariant under the twist T defined in (3), cf [BG], hence $\hat{\Phi}_\lambda = (\Phi_\lambda, \Phi_\lambda)$. Thus we are back to Theorem 4 in this case.

References

[BG] F.E. Burstall, M.A. Guest: *Harmonic two-spheres in compact symmetric spaces, revisited.* Math. Ann. 309 (1997), 541 - 572

[BP] F.E. Burstall, F. Pedit: *Harmonic maps via Adler-Kostant-Symes theory.* In: Harmonic Maps and Integrable Systems, ed. A.P. Forgy and J.C. Wood, Vieweg 1994

[DE] J. Dorfmeister, J.-H. Eschenburg: *Pluriharmonic maps, loop groups and twistor theory*, Ann. Glob. An. Geom. 24 (2003), 301 - 321

[DPW] J. Dorfmeister, F. Pedit, and H. Wu: *Weierstrass type representations of harmonic maps into symmetric spaces*, Comm. Anal. Geom. 6 (1998), 633 - 668

[E] J.-H. Eschenburg: *Pluriharmonic maps, twisted loops and twistors*, DMV-Jahresberichte 106 (2004), 39 - 48

[ET1] J.-H. Eschenburg, R. Tribuzy: *Existence and uniqueness of maps into affine homogeneous spaces*, Rend. Sem. Mat. Univ. Padova 69 (1993), 11 - 18

[ET2] J.-H. Eschenburg, R. Tribuzy: *Associated families of pluriharmonic maps and isotropy*, manuscripta math. 95 (1998), 295 - 310

[OV] Y. Ohnita and G. Valli: *Pluriharmonic maps into compact Lie groups and factorization into unitons*, Proc. Lond. Math. Soc. 61 (1990), 546 - 570

[P] K. Pohlmeyer: *Integrable Hamiltonian systems amd interactions through quadratic constraints.* Comm. Math Phys. 46 (1976), 207 - 221

[U] K. Uhlenbeck: *Harmonic maps into Lie groups (classical solutions of the chiral model)*, J.Diff.Geom **30** (1989), 1 - 50

[ZM] V.E. Zakharov and A.V. Mikhailov: *Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method.* Soviet Phys. JETP 47 (1978), 1017 - 1027

[ZS] V.E. Zakharov and A.B. Shabat: *Integration of non-linear equations of mathematical physics by the method if inverse scattering II.* Funct. Anal. Appl. 13 (1979), 13 - 22

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