GEOMETRY OF OCTONIONS

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Abstract. The simple classical compact groups SO_n, SU_n, Sp_n are related to the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ (reals, complex numbers, quaternions). The exceptional groups G_2, F_4, E_6, E_7, E_8 are somehow related to the remaining normed division algebra, the octonions $\mathbb{O}$, but the relation is not quite easy to understand, except for $G_2 = \mathrm{Aut}(\mathbb{O})$.

We will start with Hurwitz' theorem stating that $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are all possible normed division algebras over the reals (construction included). Then we will turn to the octonionic projective plane and its connection to F_4 and E_6. I tell what I know about the relation to E_7 and E_8. I also explain the relation of the octonions to real Bott periodicity.

Contents

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1. The discovery of the 4 normed division algebras

From ancient times on there was a link between numbers and geometry. Numbers came into geometry as ratio of two parallel straight line segments: How often fits the smaller segment into the larger one? Around 500 B.C., students of Pythagoras noticed that sometimes such ratio cannot be represented as a ratio of two integers. Thus the idea of (positive) real numbers representing the one-dimensional continuum was born.

Around 1570, the water engineer Rafael Bombelli from Bologna discovered the complex numbers when he tried to apply Cardano's solution formula for the cubic equation $x^3 + 3ax = 2b$,

$$x = \sqrt[3]{b + \sqrt{D}} + \sqrt[3]{b - \sqrt{D}}, \quad D = a^3 + b^2.$$

There is a situation where D is no longer positive. E.g. $x^3 - 6x = 4$ has $D = -4$. Cardano's formula would give $x = \sqrt[3]{2 + \sqrt{-4}} + \sqrt[3]{2 - \sqrt{-4}}$, but this seemed to be senseless since negative numbers are never squares. But still Bombelli dared to use these "imaginary" numbers like ordinary ones and succeeded. He noticed that $2 \pm \sqrt{-4}$ was a cubic number: $(-1 \pm \sqrt{-1})^3 = 2 \pm \sqrt{-4}$. Thus $x = -1 + \sqrt{-1} + (-1) - \sqrt{-1} = -2$ which was correct: $(-2)^3 - 6 \cdot (-2) = 4$.

Yet imaginary numbers remained mysterious for several centuries to come. Even C.F. Gauß in his Ph.D. thesis on the Fundamental Theorem of Algebra avoided them as late as 1799. But around that time, two amateur mathematicians found a geometric model for these numbers, the Danish surveyor Caspar Wessel (1797)[1] and independently the French accountant Jean-Robert Argand (1806).[2] While the reals correspond to the line, the complex numbers correspond to the plane with the imaginary numbers sitting on the vertical axis. Of course this was known also to Gauß but as in other cases he was too cautious to publish it.

In 1835 William Rowan Hamilton,[3] an Irish mathematician and astronomer, discovered independently the relation between complex numbers and planar geometry. But he wanted to go beyond. In the words

[1]Caspar Wessel, 1745 (Vestby, Norway) - 1818 (Copenhagen)
[2]Jean-Robert Argand, 1768 (Genf) - 1822 (Paris)
[3]Sir William Rowan Hamilton, 1805 - 1865 (Dublin)

of John Baez, a mathematical physicist who has worked with octonions:[4] "Fascinated by the relation between complex numbers and 2-dimensional geometry, he tried for many years to invent a bigger algebra that would play a similar role in 3-dimensional geometry. In modern language, it seems he was looking for a 3-dimensional normed division algebra. His quest built to its climax in October 1843. He later wrote to his son:

> Every morning in the early part of the above-cited month, on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me: 'Well, Papa, can you multiply triplets?' Whereto I was always obliged to reply, with a sad shake of the head: 'No, I can only add and subtract them'.

The problem was that there exists no 3-dimensional normed division algebra. He really needed a 4-dimensional algebra. Finally, on the 16th of October, 1843, while walking with his wife along the Royal Canal to a meeting of the Royal Irish Academy in Dublin, he made his momentous discovery:

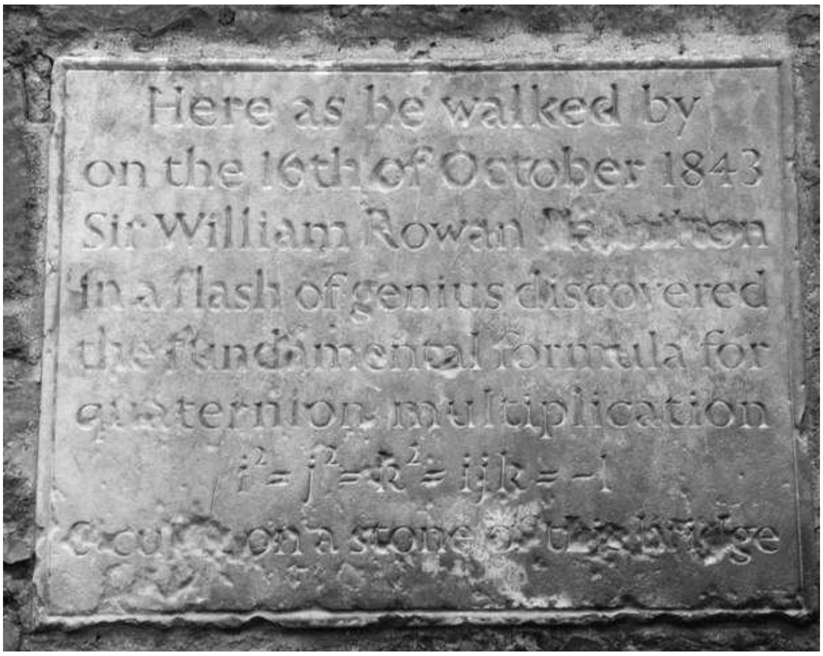
> That is to say, I then and there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations between i, j, k; exactly such as I have used them ever since.

And in a famous act of mathematical vandalism, he carved these equations into the stone of the Brougham (Broom) Bridge: $i^2 = j^2 = k^2 = ijk = -1$." John Baez further describes his finally successfull efforts in 2005 to find this bridge near Dublin and to photograph the famous plate with the inscription:

> Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication $i^2 = j^2 = k^2 = ijk = -1$ & cut it on a stone of this bridge.

The Irish mathematicians celebrate this event on the 16th october of each year at this place near Dublin.

[4]http://math.ucr.edu/home/baez/octonions/node24.html, see also https://en.wikipedia.org/wiki/John_C._Baez



These equations define a multiplication of the four basis vectors $1, i, j, k$ and thus an algebra structure on $\mathbb{R}^4$. This is called *quaternion algebra*, denoted by the symbol $\mathbb{H}$ in honour of Hamilton. However, he was not the first to use quaternions. A few years before, 1840, the French banker and mathematician Benjamin Olinde Rodrigues[5] had already introduced them in order to parametrize rotations in 3-space, but he did not introduce the multiplicative structure, and his discovery was not very well known at that time.

Remark. A faithful representation of the algebra $\mathbb{H}$ is the matrix algebra

$$\mathbb{R} \cdot SU_2 = \{\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right) : a, b \in \mathbb{C}\}$$

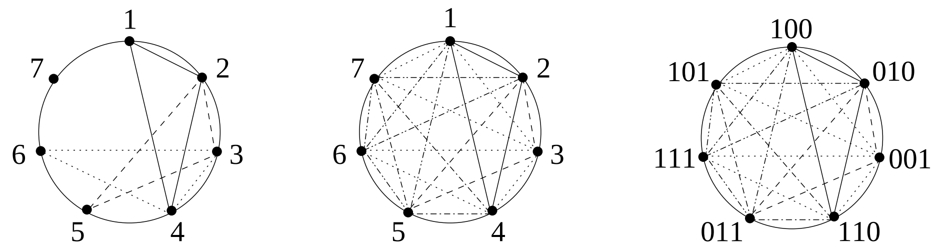
(similar to $\mathbb{C} = \mathbb{R} \cdot SO_2 = \{\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) : a, b \in \mathbb{R}\}$) where i, j, k are represented by the Pauli spin matrices $\left(\begin{smallmatrix} -i & \\ & i \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} & i \\ i & \end{smallmatrix}\right)$. This shows the associativity of $\mathbb{H}$, and further that every nonzero element $a \in \mathbb{H}$ has an inverse a^{-1}. Thus $\mathbb{H}$ is a *division algebra* which means that any linear equation $ax = b$ or $xa = b$ with given $a, b \in \mathbb{H}$ and $a \neq 0$ has a unique solution $x = a^{-1}b$ or $x = ba^{-1}$.

[5]Benjamin Olinde Rodrigues, 1795 (Bordeaux) - 1851 (Paris)

We keep citing John Baez from an 2004 interview:[6] "Among the mathematicians working in this area was John Graves,[7] a friend of Hamilton. It was Graves who had first got Hamilton interested in the problem of describing the complex numbers as pairs of ordinary numbers. When Hamilton told Graves about the quaternions, Graves' immediate question was: if you're allowed make up a way of multiplying lists of 4 numbers, why not more? And the day after Christmas of that same year, he sent a letter to Hamilton saying he had succeeded in coming up with a number system that we now call the octonions ($\mathbb{O}$), which are lists of 8 numbers, and that he had a way of multiplying them that also allowed him to divide. 'Hamilton wrote back saying, just because it's bigger I don't know if it is better,' says Baez. 'He said something like, I have a horse with four legs, I don't know if your horse with eight legs will run twice as fast!'

Back then, Baez explains, to publish a paper you would need a member of a learned society to give a talk about it at a meeting of the society, and it would appear in the society's journal, in the transcript of the meeting. Hamilton promised Graves he would talk about octonions at a meeting of the Royal Irish Academy, but he was so excited about the quaternions he kept forgetting. But, a few years later, Arthur Cayley[8] reinvented the octonions and published before Graves ever got any credit for it! Then Hamilton gave a talk saying actually Graves thought of them first, but they wound up being called the Cayley numbers for a long time. So Graves missed out on his credit."

The algebra structure of $\mathbb{O}$ is more complicated than that of $\mathbb{H}$ since there are 7 "imaginary" directions i, j, k, l, p, q, r or $e_1, \dots, e_7$ or $1, \dots 7$ for short, and when one chooses the right ordering, the multiplication table is given by the following figure where each of the triangles 124, 235, 346, etc. denotes a quaternionic subalgebra, $e_1 e_2 = e_4$ etc.



6https://plus.maths.org/content/os/issue33/features/baez/index

7John Thomas Graves, 1806 (Dublin) - 1870 (Cheltenham, England)

8Arthur Cayley, 1821 (Richmond, Surrey, England) - 1895 (Cambridge)

The figure in the center contains all secants of the heptagon and half of all triangles with three different types of secants.[9] It is a model of the Fano projective plane $\mathbb{Z}_2\mathbb{P}^2$ with 7 points and 7 lines which are represented by the triangles, see the right figure (where, say, 110 denotes the homogeneous vector $[1,1,0] \in \mathbb{Z}_2\mathbb{P}^2$). The automorphism group of this situation is the group $GL_3(\mathbb{Z}_2)$ which is the second smallest simple finite group (with 168 elements), .

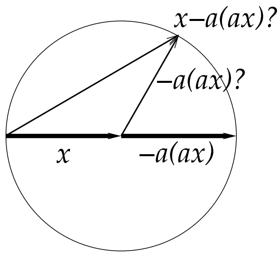
2. The normed division algebras

By an *algebra* over $\mathbb{R}$ we understand a finite dimensional real vector space $\mathbb{K}$ with a bilinear map $\mathbb{K} \times \mathbb{K} \to \mathbb{K}$, $(a,b) \mapsto a \cdot b = ab$ called *multiplication*. The algebra $\mathbb{K}$ is called a *normed algebra* or *composition algebra* if $\mathbb{K}$ carries a euclidean norm (denoted by $|a|$)[10] such that

$$|ab| = |a||b| \tag{2.1}$$

for all $a, b \in \mathbb{K}$. This is an extremely strong hypothesis. Assume further that $\mathbb{K}$ has a unit element 1 and hence a canonical embedding of $\mathbb{R} = \mathbb{R} \cdot 1$ into $\mathbb{K}$. Let $\mathbb{K}' = \mathbb{R}^\perp \subset \mathbb{K}$.

Lemma 2.1. *If $a \in \mathbb{K}'$ and $|a| = 1$, then $a(ax) = -x$ for all $x \in \mathbb{K}$, in particular $a^2 = -1$.*



Proof. $(1+a)\left((1-a)x\right) = (1-a)x + a(x - ax) = x - a(ax)$ and $|1+a||1-a||x| = 2|x|$. But from $|a(ax)| = |x|$ we see that the equality $|x - a(ax)| = 2|x|$ can hold only if the vectors x and $-a(ax)$ point in the same direction, see figure. □

Corollary 2.2. *For all $b \in \mathbb{K}'$ and $x \in \mathbb{K}$ we have $b(bx) = -|b|^2 x$ and $b^2 = -|b|^2$.*

Proof. Apply Lemma 2.1 to $a = b/|b|$ when $b \neq 0$. □

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[9] Precisely those triangles occur where short, medium and long secants appear in a clockwise ordering.

[10] "Euclidean" is unneccessary to assume since this follows from (2.1): the norm has a large group of norm-preserving linear maps given by left and right translations with unit elements in $\mathbb{K}$ – in particular it acts transitively.

Lemma 2.3. *If $a, b \in \mathbb{K}'$ and $a \perp b$, then $ab \in \mathbb{K}'$ and $ab \perp a, b$ and $ab = -ba$.*

Proof. We may assume $|a|, |b| = 1$. Since multiplications with a, b are orthogonal, we have $b \perp a \Rightarrow ab \perp a^2 = -1$ and $b \perp 1 \Rightarrow ab \perp a$, and $a \perp 1 \Rightarrow ab \perp b$. Further, $(a+b)^2 = -|a+b|^2 = -2$ by Cor. 2.2, hence

$$-2 = (a+b)^2 = a^2 + b^2 + ab + ba = -2 + (ab + ba)$$

and thus $ab + ba = 0$. □

Corollary 2.4. *Orthonormal $a, b \in \mathbb{K}'$ generate a subalgebra isomorphic to the quaternion algebra $\mathbb{H}$.*

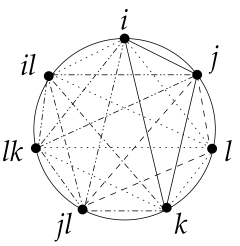
Proof. Hamilton's units $1, i, j, k$ are mapped to $1, a, b, ab$, since $(ab)^2 = -1$ and $(ab)a = -a(ab) = b$ according to Lemma 2.3. □

Lemma 2.5. *Let $\mathbb{H} \subset \mathbb{O}$ be any subalgebra isomorphic to the quaternions. If $c \perp \mathbb{H}$, then $c\mathbb{H} = \mathbb{H}c \perp \mathbb{H}$.*

Proof. For $a, b \in \mathbb{H} \cap \mathbb{K}'$ and $|a| = 1$ we have $ca \perp b$, since $(ca)a = -c \perp ba$. □

Lemma 2.6. *If $a, b, c \in \mathbb{K}'$ are orthogonal and $c \perp ab$ (a so called Cayley triple), then $(ab)c = -a(bc)$.*

Proof. According to Lemma 2.5, we also have $bc \perp a$ and therefore $a(bc) = -(bc)a = (cb)a$. Hence we have to show $(ab)c = -(cb)a$. On the one hand, $((a+c)b)(a+c) = (ab+cb)(a+c) = (ab)a + (cb)a + (ab)c + (cb)c = 2b + (cb)a + (ab)c$, on the other hand using Corollary 2.2 we have $((a+c)b)(a+c) = |a+c|^2 b = 2b$, hence $(cb)a + (ab)c = 0$. □



Remark. This figure shows the multiplication table (the "clock" on page 5) in terms of an orthonormal Cayley triple, i, j, l where $ij = k$. In fact,

$$\begin{array}{ccccccccc} k(jl) & = & -(kj)l & = & il, & & & & \\ (jl)(lk) & = & -((jl)l)k & = & jk & = & i, & & \\ (lk)(il) & = & (kl)(li) & = & -((kl)l)i & = & ki & = & j, \\ (il)i & = & -i(il) & = & l. & & & & \end{array} \tag{2.2}$$

Lemma 2.7. *Orthonormal $c, d \in A'$ with $c \perp \mathbb{H}$ and $d \perp (\mathbb{H} + c\mathbb{H})$ do not exist.*

Proof. Let $a, b \in A' \cap \mathbb{H}$ be orthonormal. Then on the one hand,

$$a(b(cd)) = -a((bc)d) = (a(bc))d\,, \tag{A}$$

since the triples (b, c, d) and (a, bc, d) are anti-associative, according to Lemma 2.6. But on the other hand:

$$a(b(cd)) = -(ab)(cd) = ((ab)c)d = -(a(bc))d\,, \tag{B}$$

since the triples (a, b, cd), (ab, c, d) and (a, b, c) are anti-associative. But (A) und (B) contradict each other! $\square$

Now we have proved:

Theorem 2.8. *(A. Hurwitz 1898)*[11] $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ *are the only normed algebras, and* $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$.

It remains to see that $\mathbb{K}$ is a division algebra. We first introduce the *conjugation* $\kappa : \mathbb{K} \to \mathbb{K}$. For any $a \in \mathbb{K} = \mathbb{R} \oplus \mathbb{K}'$ let $a = \alpha + a'$ be the corresponding decomposition ($\alpha \in \mathbb{R}$, $a' \in \mathbb{K}'$). We define an involution

$$\kappa : a = \alpha + a' \mapsto \bar{a} = \alpha - a' \tag{2.3}$$

(the reflection at the real axis) called *conjugation.*

Proposition 2.9. *κ is an anti-automorphism of $\mathbb{K}$, that is $\overline{ab} = \bar{b}\bar{a}$ for all $a, b \in \mathbb{K}$, and $a\bar{a} = \bar{a}a = |a|^2$.*

Proof. (A) Consider first the case $a, b \in \mathbb{K}'$. We may suppose $a \neq 0$, even $|a| = 1$. Then $b = \lambda a + b^\perp$ with $\lambda \in \mathbb{R}$ and $b^\perp \perp a$, and $ab = -\lambda + ab^\perp$ while $\bar{b}\bar{a} = ba = -\lambda + b^\perp a \overset{2.3}{=} -\lambda - ab^\perp = \overline{ab}$.
(B) In the general case $a = \alpha + a'$ and $b = \beta + b'$ we have

$$\begin{aligned}
\bar{b}\bar{a} &= \beta\alpha - \beta a' - \alpha b' + b'a', \\
ab &= \alpha\beta + \alpha b' + \beta a' + a'b', \\
\overline{ab} &= \alpha\beta - \alpha b' - \beta a' + \overline{a'b'} \overset{(A)}{=} \bar{b}\bar{a}
\end{aligned}$$

If $b = \bar{a} = \alpha - a'$, then $ab = \alpha^2 - \alpha a' + \alpha a' - a'a' = \alpha^2 + |a'|^2 = |a|^2$. $\square$

Corollary 2.10. *Any normed algebra $\mathbb{K}$ (with unit 1) is a division algebra: The solutions of the equations $ax = b$ and $xa = b$ with $a \neq 0$ are $x = a^{-1}b$ and $x = ba^{-1}$ with*

$$a^{-1} = \bar{a}/|a|^2. \tag{2.4}$$

[11] Adolf Hurwitz, 1858 (Hildesheim) - 1919 (Zürich)

Proof. $|a|^2 x = \bar{a}ax \overset{ax=b}{=} \bar{a}b \Rightarrow x = \frac{\bar{a}}{|a|^2}b$,

and $x|a|^2 = xa\bar{a} \overset{xa=b}{=} b\bar{a} \Rightarrow x = b\frac{\bar{a}}{|a|^2}$. $\square$

Remark. The product $\bar{a}ax$ (as well as $xa\bar{a}$) needs no parentheses since $\bar{a} = -a + 2\alpha$, hence $(\bar{a}a)x = -aax + 2\alpha ax = \bar{a}(ax)$.

3. The automorphism groups of the normed algebras

An *automorphism* of an $\mathbb{R}$-algebra $\mathbb{K}$ is a linear isomorphism $\phi : \mathbb{K} \to \mathbb{K}$ which preserves the product,

$$\phi(a)\phi(b) = \phi(ab). \tag{3.1}$$

For $\mathbb{K} = \mathbb{R}$, the only automorphism is the identity since the basis element $1 \in \mathbb{R}$ must be fixed. More generally, an automorphism of any algebra $\mathbb{K}$ fixes $\mathbb{R} = \mathbb{R} \cdot 1 \subset \mathbb{K}$.

For $\mathbb{K} = \mathbb{C}$, any automorphism ϕ satisfies $\phi(i) = \pm i$ since $\phi(i)^2 = \phi(-1) = -1$. Thus $\mathrm{Aut}(\mathbb{C}) = \{\mathrm{id}, \kappa\}$ where $\kappa(a) = \bar{a}$ is complex conjugation.

For $\mathbb{K} = \mathbb{H}$ and $\mathbb{K} = \mathbb{O}$, any automorphism ϕ preserves $\mathbb{K}' = \{a \in \mathbb{K} : a^2 \in \mathbb{R}_{\leq 0}\}$ (see 2.2).[12] Thus ϕ preserves the eigenspaces $\mathbb{R}$ and $\mathbb{K}'$ of the conjugation κ defined by (2.3), hence ϕ and κ commute. Consequently, $|\phi(a)|^2 = \phi(a)\kappa(\phi(a)) = \phi(a\kappa(a)) = |a|^2$, thus ϕ is orthogonal. Therefore, $\mathrm{Aut}(\mathbb{H}) \subset O_3$ and $\mathrm{Aut}(\mathbb{O}) \subset O_7$. We will compute $\mathrm{Aut}(\mathbb{K})$ by replacing it with a a submanifold of a Stiefel manifold of orthonormal frames on which it acts simply transitively.

For $\mathbb{K} = \mathbb{H}$, any orthonormal 2-frame (a, b) in $\mathbb{K}'$ can be mapped by a unique automorphism to (i, j), hence $\mathrm{Aut}(\mathbb{H})$ can be viewed as the set $V_2(\mathbb{R}^3)$ or orthonormal 2-frames in $\mathbb{K}' = \mathbb{R}^3$. Since (a, b) can be uniquely extended to the oriented 3-frame (a, b, ab), this set is the group SO_3. In fact, the unit sphere $\mathbb{S}^3 \subset \mathbb{H}$ acts on $\mathbb{H}'$ by conjugation $x \mapsto ax\bar{a}$, and the kernel of this action is precisely $\{\pm 1\}$. Thus $\mathrm{Aut}(\mathbb{H})$ is the group $SO_3 = \mathbb{S}^3/\pm$ acting on $\mathbb{K}'$ by conjugation.

The automorphism group acts also on the set of subalgebras isomorphic to $\mathbb{C}$, and the isotropy group of the standard $\mathbb{C} \subset \mathbb{H}$ corresponds to the pairs $(\pm i, b)$ with $b \perp i$ which is O_2. Thus the manifold of subalgebras isomorphic to $\mathbb{C}$ is $SO_3/O_2 = \mathbb{RP}^2$.

For $\mathbb{K} = \mathbb{O}$, we replace the 2-frames (a, b) by orthonormal Cayley triples (a, b, c) in $\mathbb{O}' = \mathbb{R}^7$. Again, there is a unique automorphism

[12] Conversely, $a \notin \mathbb{K}' \Rightarrow a^2 \notin \mathbb{R}_{\leq 0}$. In fact, let $a = \alpha + a'$ with $\alpha \neq 0$. Then $a^2 = \alpha^2 + 2\alpha a' - |a'|^2$. When $a' \neq 0$, this has a $\mathbb{K}'$-component, and when $a' = 0$, then $a^2 = \alpha^2 > 0$.

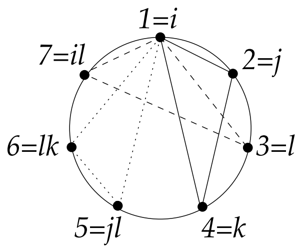
sending (a,b,c) to the standard triple (i,j,l), thus $\mathrm{Aut}(\mathbb{O})$ acts simply transitively on the set C of all Cayley triples. The map $(a,b,c)\mapsto(a,b)$ defines a fibre bundle over the Stiefel manifold $V_2(\mathbb{R}^7)$. Given a,b, the third vector c is an arbitrary unit vector perpendicular to a,b,ab. But $(a,b,ab)^\perp\cong\mathbb{R}^4$, thus the fibre (the set of all c) is a 3-sphere $\mathbb{S}^3$. Hence C is an $\mathbb{S}^3$-bundle over $V_2(\mathbb{R}^7)$ which is itself an $\mathbb{S}^5$-bundle over $\mathbb{S}^6$ (the unit sphere in $\mathbb{O}'$). In particular, $\mathrm{Aut}(\mathbb{O})$ is connected and has dimension $6+5+3=14$. We also see that $\mathrm{Aut}(\mathbb{O})$ contains the finite group $GL_3(\mathbb{Z}_2)$ which permutes the Cayley triples among the basis elements (where the standard Cayley triple (i,j,l) is identified with $(100,010,001)$, see right figure on page 5).

The group $\mathrm{Aut}(\mathbb{O})$ acts also on the set of all subalgebras isomorphic to $\mathbb{H}$, and the isotropy group $K=\mathrm{Aut}(\mathbb{O})_{\mathbb{H}}$ consists of the triples (a,b,c) with $a,b\in\mathbb{H}$, that is $(a,b)\in V_2(\mathbb{R}^3)=SO_3$. In particular, K acts transitively on the set of unit vectors $c\perp\mathbb{H}$, thus on $\mathbb{S}^3\subset\mathbb{H}^\perp$. The stabilizer in K for the standard element $c=l$ is $\mathrm{Aut}(\mathbb{H})=SO_3$, thus $\mathbb{S}^3=K/SO_3$ whence $K=SO_4$ with its standard action on $\mathbb{H}^\perp=\mathbb{R}^4$. In other words, each automorphism $\mathrm{Ad}(q)$ of $\mathbb{H}$ is extended to an automorphism ϕ of $\mathbb{O}$ by putting $\phi(l)=pl$ for an arbitrary $p\in\mathbb{H}$, $|p|=1$, and hence for any $a,b\in\mathbb{H}$ we have

$$\phi(a+bl)=\phi(a)+\phi(b)\phi(l)=qaq^{-1}+(qbq^{-1})(pl). \tag{3.2}$$

Choosing $q,p\in\mathbb{S}^1\subset\mathbb{C}\subset\mathbb{H}$, we find a maximal torus T of $\mathrm{Aut}(\mathbb{O})$ within $K\cong SO_4$.

The next figure shows the three triangles ("lines", quaternionic subalgebras) between the basis elements with 1 $(=i)$ as one vertex.



The end point pairs of the edges 24, 37, 56 opposite to 1 span three 2-dimensional T-modules since the pairs are preserved by left and right multiplications by i. Putting

$$J_{ab}: e_b\mapsto e_a,\ e_a\mapsto -e_b,\ e_c\mapsto 0 \text{ for all } c\neq a,b \tag{3.3}$$

(with $a,b,c\in\{1,\dots,7\}$), the Lie algebra of T is

$$\mathfrak{t}=\{\alpha J_{24}+\beta J_{37}+\gamma J_{56}:\alpha+\beta+\gamma=0\}. \tag{3.4}$$

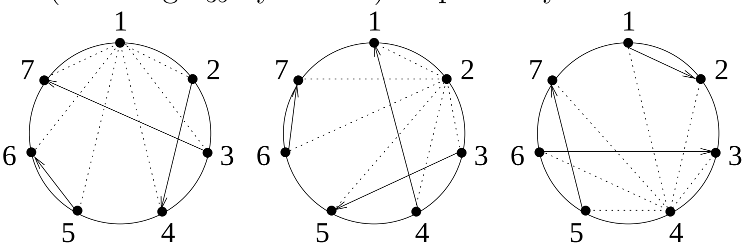
In fact, T consists of automorphisms $\phi = \phi_{st}$ of type (3.2) with $q = e^{is/2}$ and $p = e^{it}$ whose infinitesimal generators X, Y are $X(a + bl) = \frac{1}{2}([i, a] + [i, b]l)$ and $Y(a + bl) = b(il)$. Thus the endpoint pairs of the edges opposite to i are mapped by X and Y as follows:

$$\begin{array}{rcl}(j,k),\ (l,il),\ (jl,lk) & \overset{X}{\mapsto} & (k,-j),\ (0,0),\ (kl,-lj)\\ (j,k),\ (l,il),\ (jl,lk) & \overset{Y}{\mapsto} & (0,0),\ (-il,l),\ (kl,-lj)\end{array}$$

Hence $X = J_{56} - J_{24} \in \mathfrak{t}$ and $Y = J_{56} - J_{37} \in \mathfrak{t}$, see (3.4), and $\mathfrak{t} = \mathrm{span}\,(X, Y)$ since X, Y form a basis of $\mathfrak{t}$.

Similarly, we define the subspaces $V_a \subset \mathfrak{aut}(\mathbb{O})$ replacing 1 by any $a \in \{1, \dots, 7\}$. This defines a decomposition of the Lie algebra of $\mathrm{Aut}(\mathbb{O})$ into seven 2-dimensional subspaces. The action of T or $\mathfrak{t} = V_1$ leaves $V_a + V_b$ invariant for every pair $(a, b) \in \{(2, 4), (3, 7), (5, 6)\}$, and its complexification is decomposed into the root spaces for $\mathfrak{t}$ (see [7], page 29).[13]

In fact, recall that $[J_{ab}, J_{cd}] = 0$ when a, b, c, d are distinct (as in $\mathfrak{so}_4$), and $[J_{ab}, J_{bc}] = J_{ac}$ while $[J_{ab}, J_{ac}] = -J_{bc}$ (as in $\mathfrak{so}_3$). Hence the complex matrices $J_{ac} + iJ_{bc}$ and $J_{bc} - iJ_{ac}$ are eigenvectors for $\mathrm{ad}(J_{ab})$ with eigenvalue i since $[J_{ab}, J_{bc} - iJ_{ac}] = J_{ac} + iJ_{bc} = i(J_{bc} - iJ_{ac})$. E.g. consider V_2 and V_4 which are spanned by the differences of $35, 41, 67$ and $57, 63, 12$ (denoting J_{35} by 35 etc.) respectively.



V_1 V_2 V_4

Thus we have the following i-eigenvectors,

ad(24)	ad(37)	ad(56)	
$21 + 41\,i$	$35 + 75\,i$	$53 + 63\,i$	
$= i(\underline{41 + 12\,i})$	$= \underline{35 - 57\,i}$	$= -(\underline{35 + 36\,i})$	(3.5)
	$36 + 76\,i$	$57 + 67\,i$	
	$= -i(\underline{67 + 36\,i})$	$= i(\underline{67 - 57\,i})$	

[13]Freudenthal uses a different counting for the seven basis vectors. Our notation is changed into his by the permutation $1234567 \mapsto 1342765$. Moreover, instead of $V_2 + V_4$ he considers $V_5 + V_6$ (in our notation).

while the complex conjugates belong to the eigenvalue $-i$. Let

$$\begin{array}{rclcl}
v_- &=& 35-57\,i \;-\; (67+36\,i) &=& 35-36\,i \;-\; (67+57\,i)\\
v_+ &=& 35-57\,i \;+\; (67+36\,i) &=& 35+36\,i \;+\; (67-57\,i)\\
v_o &=& 41+12\,i &&
\end{array}$$

Then

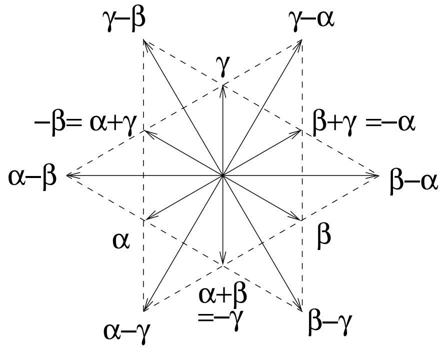
$$\begin{array}{rcl}
[\alpha\cdot 24+\beta\cdot 37+\gamma\cdot 56, v_-] &=& i(\beta-\gamma)v_-\\
[\alpha\cdot 24+\beta\cdot 37+\gamma\cdot 56, v_+] &=& i(\beta+\gamma)v_+\\
[\alpha\cdot 24+\beta\cdot 37+\gamma\cdot 56, v_o] &=& i\alpha v_o.
\end{array}$$

Hence $v_- \in (V_2+V_4)\otimes\mathbb{C}$ is an eigenvector of $\mathrm{ad}(\alpha\cdot 24+\beta\cdot 37+\gamma\cdot 56)$ for the eigenvalue $\beta-\gamma$. On the other hand, $v_+ - 2v_o \in (V_2+V_4)\otimes\mathbb{C}$ is an eigenvector for the eigenvalue $\beta+\gamma$, provided that $\alpha+\beta+\gamma=0$:

$$\begin{array}{rcl}
[\alpha\cdot 24+\beta\cdot 37+\gamma\cdot 56\,,\ -2v_o+v_+] &=& -2i\alpha v_o + i(\beta+\gamma)v_+\\
&=& i(\beta+\gamma)(-2v_o+v_+)
\end{array}$$

when $\alpha=-(\beta+\gamma)$. Analogous statements hold for the complex conjugate vectors. Now $(V_2+V_4)\otimes\mathbb{C}$ is decomposed into eigenspaces of $\mathrm{ad}(\alpha\cdot 24+\beta\cdot 37+\gamma\cdot 56)$ with eigenvalues (roots) $\pm(\beta-\gamma)$, $\pm(\beta+\gamma)$.

We may use symmetry to compute the remaining roots. The group of automorphisms of $\mathbb{O}$ preserving the basis $e_1,\dots,e_7$ of $\mathbb{O}'$ is $GL_3(\mathbb{Z}_2)\subset \mathrm{Aut}(\mathbb{O})$. The stabilizer of e_1 inside this group permutes the oriented secants $\{24,37,56\}$.[14] Thus it preserves the torus T and permutes α,β,γ, hence the full set of roots is $\pm\alpha\pm\beta$, $\pm\beta\pm\gamma$, $\pm\gamma\pm\alpha$ where $\alpha+\beta+\gamma=0$. Now we have proved that the root system has Dynkin type G_2, see subsequent figure.



The maximal torus T is a maximal abelian subgroup of $\mathrm{Aut}(\mathbb{O})$, but not the only one (up to conjugacy). There is a maximal abelian subgroup of type $(\mathbb{Z}_2)^3$, a maximal "2-group", which is not contained in any maximal torus of $\mathrm{Aut}(\mathbb{O})$. In fact, each of the seven secant triangles in

[14]This is the subgroup $GL_2(\mathbb{Z}_2)\subset GL_3(\mathbb{Z}_2)$ with the embedding $A\mapsto \left(\begin{smallmatrix}1&\\&A\end{smallmatrix}\right)$, $A\in GL_2(\mathbb{Z}_2)=\{\left(\begin{smallmatrix}1&\\&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&1\\&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&1\\1&\end{smallmatrix}\right),\left(\begin{smallmatrix}1&\\1&1\end{smallmatrix}\right),\left(\begin{smallmatrix}&1\\1&1\end{smallmatrix}\right),\left(\begin{smallmatrix}&1\\1&\end{smallmatrix}\right)\}\cong S_3$.

the figure on page 5 defines a quaternionic subalgebra $\mathbb{H}_a$, $a = 1, \dots, 7$, and the reflection at $\mathbb{H}_a$ is an automorphism. E.g. the reflection at $\mathbb{H}_1$ maps the Cayley triple (i, j, l) onto the Cayley triple $(i, j, -l)$; it corresponds to $-I \in SO_4 \subset \mathrm{Aut}(\mathbb{O})$. Thus we have seven reflections along certain coordinate 3-planes in $\mathbb{O}' = \mathbb{R}^7$. They commute and define a subgroup isomorphic to $(\mathbb{Z}_2)^3$ which is maximal abelian, see Borel-Hirzebruch [4], p. 531. The subgroup $GL_3(\mathbb{Z}_2) \subset \mathrm{Aut}(\mathbb{O})$ permutes the seven quaternionic subalgebras, and it is the normalizer of $(\mathbb{Z}_2)^3$ in $\mathrm{Aut}(\mathbb{O})$, similar to the Weyl group which is the normalizer (made effective) of the maximal torus. This is essential for computing the $\mathbb{Z}_2$-cohomology (Borel) and the Stiefel-Whitney classes (see [4] and references therein).

4. Hermitian 3×3-matrices over $\mathbb{K}$

Grassmannians over a division algebra $\mathbb{K}$ can be represented in the space of hermitian matrices over $\mathbb{K}$ where each subspace $E \subset \mathbb{K}^n$ (an element of the Grassmannian) is mapped to the orthogonal projection onto E. This is the hermitian matrix P over $\mathbb{K}$ with $P^2 = P$ and image E. This construction still works for $\mathbb{K} = \mathbb{O}$.

Let $M_n = \mathbb{K}^{n \times n}$ be the space of $n \times n$-Matrices over $\mathbb{K}$. It carries the standard inner product

$$\langle X, Y \rangle = \mathrm{Re\ trace}\,(X^* Y) = \sum \bar{X}_{ij} Y_{ij} = \langle Y, X \rangle.$$

For any $A = (A_{ij}) \in M_n$ we define the adjoint matrix $A^* \in M_n$ with

$$A^*_{ij} = \kappa(A_{ji}) = \overline{A_{ji}}.$$

The map $X \mapsto X^*$ is an anti-automorphism of the matrix algebra M_n. In fact, for any $A, B \in M_n$ we have $(B^* A^*)_{ik} = \sum_j B^*_{ij} A^*_{jk} = \sum_j \kappa(B_{ji})\kappa(A_{kj}) = \sum_j \kappa(A_{kj} B_{ji}) = \kappa(AB)_{ki} = (AB)^*_{ik}$.

Further, for all $X, Y, Z \in M_n$ we have

$$\langle XY, Z \rangle = \mathrm{Re\ trace}\,(XY)^* Z = \mathrm{Re\ trace}\ Y^* X^* Z = \langle Y, X^* Z \rangle. \tag{4.1}$$

In fact, recall that $\mathrm{Re\ trace}\,(ABC)$ is associative:

$$((AB)C)_{ii} = \sum (A_{ij} B_{jk}) C_{ki} \text{ and } (A(BC))_{ii} = \sum A_{ij}(B_{jk} C_{ki}).$$

Thus the difference is a sum of associators $[A_{ij}, B_{jk}, C_{ki}]$ where

$$[a, b, c] = (ab)c - a(bc) \tag{4.2}$$

for all $a, b, c \in \mathbb{K}$. The associator is always imaginary[15] and anti-symmetric (since it vanishes whenever two arguments are equal). In particular, the real part of the associator vanishes.

We let

$$H_n = \{X \in M_n : X^* = X\} \text{ and } A_n = \{X \in M_n : X^* = -X\}$$

be the subspaces of hermitian and anti-hermitian matrices. We let

$$\mathsf{G}_k(\mathbb{K}^n) = \{P \in H_n : P^2 = P,\ \text{trace } P = k\} \subset H_n \tag{4.3}$$

be the Grassmannian of k-dimensional subspaces of $\mathbb{K}^n$. If $\mathbb{K} = \mathbb{O}$, there are no subspaces (submodules) since $\mathbb{O}^n$ is not a module over $\mathbb{O}$, by lack of associativity. But still

$$\mathbb{OP}^2 := \mathsf{G}_1(\mathbb{O}^3) \subset H_3(\mathbb{O}) \tag{4.4}$$

is defined in the sense of (4.3), called the *octonionic projective plane.*

The vector space H_n becomes an algebra with the commutative product

$$X \circ Y = \frac{1}{2}(XY + YX).$$

In fact, $X \circ Y \in H_n$ since $(XY + YX)^* = YX + XY$. Further, there is an inner product and a 3-form

$$(X, Y, Z) = \langle X \circ Y, Z\rangle = \langle Z, X \circ Y\rangle$$

which is symmetric: it is invariant under transposition of X and Y, and it is invariant under cyclic permutations (using $\text{Re trace}\,(UV) = \text{Re trace}\,(VU)$):

$$\begin{aligned} 2(X, Y, Z) &= \text{Re trace}\,(ZXY + \underline{ZY}X) \\ &= \text{Re trace}\,(ZXY + X\underline{ZY}) \\ &= 2(Z, X, Y). \end{aligned}$$

5. The automorphism group of $H_3(\mathbb{O})$

Again we are interested in the automorphism group of this structure, more precisely in the group $\text{Aut}(H_n)$ of orthogonal automorphisms of H_n. Clearly, this group leaves $\mathsf{G}_k(\mathbb{K}^n) \subset H_n$ invariant since the defining equations $P^2 = P$ and $k = \text{trace } X = \langle X, I\rangle$ are invariant under orthogonal automorphisms. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, this group is well known: its identity component is the unitary group

$$U_n(\mathbb{K}) = \{g \in \mathbb{K}^{n\times n} : g^*g = I\}$$

[15] $\langle (ab)c, 1\rangle = \langle ab, \bar{c}\rangle$ while $\langle a(bc), 1\rangle = \langle bc, \bar{a}\rangle = \langle b, \bar{a}\bar{c}\rangle = \langle ab, \bar{c}\rangle$ when a, b, c are unit octonions.

(devided by its center) acting on $H_n(\mathbb{K})$ by conjugation, note that $gXg^{-1} = gXg^* \in H_n$ for all $X \in H_n$. However, for $\mathbb{K} = \mathbb{O}$ this group is very small in general (it always contains $\mathrm{Aut}(\mathbb{O})$ and the permutation group S_n), but for $n = 3$ it is large: it is the exceptional group of Dynkin type F_4 as we will see.

Also A_n is an algebra with the anti-commutative product

$$[AB] = AB - BA.$$

Further any $A \in A_n$ acts on H_n by

$$[AX] = AX - XA \tag{5.1}$$

since $(AX - XA)^* = -XA + AX = AX - XA$, hence $[AX] \in H_n$ for all $X \in H_n$. If $\mathbb{K}$ is associative, this action is by *derivations*. Recall that a derivation of an algebra $\mathcal{A}$ is a linear map $\delta : \mathcal{A} \to \mathcal{A}$ with

$$\delta(ab) = (\delta a)b + a(\delta b) \tag{5.2}$$

for all $a, b \in \mathcal{A}$. Derivations are derivatives of one-parameter groups of automorphisms: if ϕ_t, $t \in \mathbb{R}$, is a family of automorphisms of $\mathcal{A}$ with $\phi_0 = I = \mathrm{id}$ and $\frac{d}{dt}\big|_{t=0} \phi_t = \delta$, then differentiation of $\phi_t(ab) = \phi_t(a)\phi_t(b)$ at $t = 0$ gives (5.2). Thus the derivations form the Lie algebra (the tangent space at I) of the group $\mathrm{Aut}(\mathcal{A})$.

Lemma 5.1. *If $\mathbb{K}$ is associative, then A_n acts by derivations on H_n.*

Proof.

$$\begin{aligned}
2\,\mathrm{ad}_A(X \circ Y) &= [A, (XY + YX)] \\
&= A(XY) + A(YX) - (XY)A - (YX)A, \\
2\,(\mathrm{ad}_A X \circ Y + X \circ \mathrm{ad}_A Y) &= [AX]Y + Y[AX] + X[AY] + [AY]X \\
&= \underline{(AX)Y} - (XA)Y + Y(AX) - \underline{Y(XA)} \\
&\quad + X(AY) - \underline{X(YA)} + \underline{(AY)X} - (YA)X.
\end{aligned}$$

In fact, the underlined terms are those of $\mathrm{ad}_A(X \circ Y)$, and the remaining ones cancel each other (provided that one can neglect parentheses). $\square$

This claim is true also because in the associative case A_n acting by (5.1) is the Lie algebra of the group $U_n(\mathbb{K})$ acting by conjugation as a group of automorphisms of H_n.

Nothing of this remains true for $\mathbb{K} = \mathbb{O}$. Invertible matrices over $\mathbb{O}$ do not even form a group, by lack of associativity, and A_n is not a Lie algebra (Jacobi identity fails). Moreover, in the above proof associativity was often used. But still the claim remains true for $n = 3$; this is one of the miracles of this theory. We only have to restrict our attention to

$$A_n^o := \{A \in A_n : \mathrm{trace}\ A = 0\}.$$

Theorem 5.2. *$A_3^o(\mathbb{O})$ acts on $H_3(\mathbb{O})$ by skew-adjoint derivations, that is:* $\mathrm{ad}(A) : X \mapsto [AX]$ *is a skew adjoint derivation for any $A \in A_3^o$.*

The main observation for the proof (see [Freudenthal]) is the following

Lemma 5.3. *For every $X \in H_3$ and $A \in A_3$ with* trace $A = 0$ *we have*

$$\langle [AX], X^2 \rangle = 0. \tag{5.3}$$

Proof of Theorem 5.2. Polarizing (5.3)[16] we obtain the derivation property, using the symmetry of $\tau(X, Y, Z) = \langle X \circ Y, Z \rangle$:

$$\begin{aligned} 0 &= \langle [AX], Y \circ Z \rangle + \langle [AY], Z \circ X \rangle + \langle [AZ], X \circ Y \rangle \\ &= \langle [AX] \circ Y, Z \rangle + \langle X \circ [AY], Z \rangle + \langle X \circ Y, [AZ] \rangle \\ &= \langle [AX] \circ Y + X \circ [AY] - [A, X \circ Y]\ ,\ Z \rangle. \end{aligned}$$

In the last step we have used that $\mathrm{ad}(A)$ is skew adjoint. Proof:
$\langle [AW], Z \rangle + \langle W, [AZ] \rangle = \mathrm{Re\ trace}\,(\underline{AWZ} - WAZ + WAZ - \underline{WZA}) = 0$

Proof of Lemma 5.3. We show for all $X \in H_3$

$$X^2X - XX^2 = aI, \quad a \in \mathbb{K}' = \{x \in \mathbb{K};\ x \perp 1\}. \tag{5.4}$$

Then the claim follows:

$$\langle AX, X^2 \rangle \overset{(4.1)}{=} \langle A, X^2X \rangle \overset{(5.4)}{=} \langle A, XX^2 \rangle \overset{(4.1)}{=} \langle XA, X^2 \rangle$$

where we have used $\langle A, aI \rangle = \langle \bar{a}A, I \rangle = -a$ trace $A = 0$. To prove (5.4) we consider the matrix

$$D = X^2X - XX^2$$

with entries

$$d_{ij} = \sum_{kl} ((x_{ik}x_{kl})x_{lj} - x_{ik}(x_{kl}x_{lj})) = \sum_{kl} [x_{ik}, x_{kl}, x_{lj}]$$

(see (4.2)). When one of the three coefficients x_{ik}, x_{kl}, x_{lj} is real or if two of them are conjugate, then $[x_{ik}, x_{kl}, x_{lj}] = 0$. Since x_{kk} is real and $x_{kl} = \bar{x}_{lk}$, both index triples (i, k, l) and (k, l, j) must be pairwise distinct when $d_{ij} \neq 0$. But there are only three indices, hence $i = j$. Thus $d_{ij} = 0$ for $i \neq j$ and

$$d_{ii} = [x_{ik}, x_{kl}, x_{li}] + [x_{il}, x_{lk}, x_{ki}]$$

with $\{i, j, k\} = \{1, 2, 3\}$, e.g.

$$d_{11} = [x_{12}, x_{23}, x_{31}] + [x_{13}, x_{32}, x_{21}]$$

[16] The cubic form $f(X) = \langle [AX], X \circ X \rangle$ is the diagonal of the symmetric 3-form (its 3rd derivative) $\tau(X, Y, Z) = \frac{1}{3}(\langle [AX], Y \circ Z \rangle + \langle [AY], Z \circ X \rangle + \langle [AZ], X \circ Y \rangle)$, that means $f(X) = \tau(X, X, X)$. The 3-form τ is symmetric since it is invariant under cyclic permutation and under the exchange of Y and Z.

Since the associator is antisymmetric and therefore invariant under cyclic permutations, we have $d_{11} = d_{22} = d_{33}$. Each associator is imaginary, so (5.4) is proved and the claim follows. $\square$

Remark. For the associator of any $a, b, c \in \mathbb{O}$ we have

$$\overline{[a,b,c]} = \overline{(ab)c} - \overline{a(bc)} = \bar{c}(\bar{b}\bar{a}) - (\bar{c}\bar{b})\bar{a} = -[\bar{c}, \bar{b}, \bar{a}].$$

Hence $[x_{13}, x_{32}, x_{21}] = [\bar{x}_{31}, \bar{x}_{23}, \bar{x}_{12}] = -\overline{[x_{12}, x_{23}, x_{31}]}$ and therefore

$$d_{11} = [x_{12}, x_{23}, x_{31}] - \overline{[x_{12}, x_{23}, x_{31}]} = 2[x_{12}, x_{23}, x_{31}]$$

since $[x_{12}, x_{23}, x_{31}] \in \mathbb{O}'$.

One might wonder why only trace-zero matrices in A_n are considered. What about the complement, the scalar matrices $A = aI$, $a \in \mathbb{K}'$? Obviously, equation (5.3) remains true for $A = aI$ when $\mathbb{K}$ is associative since then $XX^2 = X^2X$. But then $\mathrm{ad}(aI)$ is a derivation of $\mathbb{K}$ itself (the derivative of the conjugation $\phi_t(x) = e^{ta}xe^{-ta}$). Clearly automorphisms of $\mathbb{K}$, applied to each coefficient of a matrix, are also automorphisms of H_n, and similar for derivations. But when $\mathbb{K} = \mathbb{O}$, $\mathrm{ad}(aI)$ is no longer a derivation (since (5.3) is not true anymore), and in general, conjugations are not automorphisms, by lack of associativity. However it remains true that $\mathrm{ad}(A_3^o)$ is a complement of $\mathfrak{aut}(\mathbb{K})$ in the Lie algebra of $\mathrm{Aut}(H_3)$. First recall a well known fact for the associative case, the *Jacobi identity*:

Lemma 5.4. *If $\mathbb{K}$ is associative, then for all $A, B \in A_n$*

$$\mathrm{ad}[A,B] = [\mathrm{ad}(A), \mathrm{ad}(B)]. \tag{5.5}$$

Proof.

$$\begin{aligned}
[[AB]X] &= ABX - BAX - XAB + XBA, \\
[A[BX]] - [B[AX]] &= ABX - \underline{AXB} - BAX + \underline{BXA} \\
&\quad - \underline{BXA} + XBA + \underline{AXB} - XAB.
\end{aligned}$$

The underlined summands cancel each other, and the remaining terms are those of the first line. $\square$

In $A_3(\mathbb{O})$, (5.5) is no longer true. But for any $A, B \in A_n^o$ we have $[\mathrm{ad}(A), \mathrm{ad}(B)] \in \mathfrak{aut}(H_n)$ and $\mathrm{ad}([A,B]^o) \in \mathfrak{aut}(H_n)$ where C^o denotes the trace-free part of $C \in A_n$, that is $C^o = C - cI$ with $c = \mathrm{trace}\, C/n$. The proof of Lemma 5.5 shows that

$$\delta = [\mathrm{ad}(A), \mathrm{ad}(B)] - \mathrm{ad}[A,B]^o$$

satisfies $\delta(X) = 0$ for any real $X \in H_n$ (only products of at most two non-real octonionic factors occur), thus $\delta \in \mathfrak{aut}(\mathbb{K})$ by the next proposition. We have no longer $[\mathrm{ad}(A), \mathrm{ad}(B)] = \mathrm{ad}([A,B])$ if $\mathbb{K} = \mathbb{O}$

(this is the Jacobi identity which needs associativity of $\mathbb{K}$), but still $[\mathrm{ad}(A), \mathrm{ad}(\mathfrak{B})] \equiv \mathrm{ad}[A,B]^o \mod \mathfrak{aut}(\mathbb{K})$. Hence $\mathfrak{aut}(H_n) = \mathrm{ad}(A_n^o) \oplus \mathfrak{aut}(\mathbb{K})$ (where $n = 3$ when $\mathbb{K} = \mathbb{O}$). We have $\dim A_3^o(\mathbb{O}) = 3 \cdot 8 + 2 \cdot 7 = 38$ and $\dim \mathrm{Aut}(\mathbb{O}) = 14$, thus $\dim \mathrm{Aut}(H_3(\mathbb{O})) = 38 + 14 = 52$, see 5.7 below. This is the dimension of the simple group with Dynkin diagram F_4.

Proposition 5.5. *An automorphism ϕ of $H_n(\mathbb{K})$ fixing $H_n(\mathbb{R})$ lies in* $\mathrm{Aut}(\mathbb{K})$.

Lemma 5.6. *Let $H_{kl} \subset H_n(\mathbb{K})$ be the set of matrices X with $X_{ij} = 0$ when $\{ij\} \not\subset \{kl\}$. Let $P_j \in H_n(\mathbb{R})$ with $(P_j)_{jj} = 1$ and all other entries $(P_j)_{kl} = 0$. Then*

$$H_{kl} = \{X \in H_n : X \circ P_j = 0 \textit{ for all } j \neq k,l\}. \tag{5.6}$$

Proof. We put $X = \begin{pmatrix} \alpha & v^* \\ v & X' \end{pmatrix}$ with $\alpha \in \mathbb{R}$ and $v \in \mathbb{K}^{n-1}$ and $X' \in H_{n-1}$. Then

$$2X \circ P_1 = 2 \begin{pmatrix} \alpha & v^* \\ v & X' \end{pmatrix} \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2\alpha & v^* \\ v & 0 \end{pmatrix}.$$

Thus $X \circ P_1 = 0 \iff \alpha = 0$ and $v = 0$. Similarly, $X \circ P_j = 0$ if and only if the j-th row and column are zero. Thus only the k-th and l-th row and column survive if $X \circ P_j$ for all $j \neq k,l$. $\square$

Proof of Proposition 5.5. Let ϕ be an automorphism of $H_n(\mathbb{K})$ fixing $H_n(\mathbb{R})$. Since the characterization (5.6) is invariant under such automorphisms, ϕ leaves each H_{kl} invariant ($k \neq l$), and the same holds for $H'_{kl} := H_{kl} \cap H_n(\mathbb{R})^\perp = \mathbb{K}' \cdot J_{kl}$, see (3.3). Thus for any pair $k \neq l$ there is a map $\varphi_{kl} \in O(\mathbb{K})$ fixing 1 such that $\phi(X)_{kl} = \varphi_{kl}(X_{kl})$ for any $X = (X_{kl}) \in H_n(\mathbb{K})$. We apply this to $X = \begin{pmatrix} & x & \\ -x & & \\ & & \end{pmatrix}$ and $Y = \begin{pmatrix} & & \\ & & y \\ & -y & \end{pmatrix}$ with $x, y \in \mathbb{K}'$. Then we have (with new notation $\varphi, \varphi', \varphi''$ instead φ_{kl})

$$\begin{aligned} \phi(X \circ Y) &= \phi\left(\begin{pmatrix} & x & \\ -x & & \\ & & \end{pmatrix} \circ \begin{pmatrix} & & \\ & & y \\ & -y & \end{pmatrix}\right) &= \begin{pmatrix} & & \varphi(xy) \\ & & \\ \varphi(yx) & & \end{pmatrix}, \\ \phi(X) \circ \phi(Y) &= \begin{pmatrix} & \varphi' x & \\ -\varphi' x & & \\ & & \end{pmatrix} \circ \begin{pmatrix} & & \\ & & \varphi'' y \\ & -\varphi'' y & \end{pmatrix} &= \begin{pmatrix} & & \varphi'(x)\varphi''(y) \\ & & \\ \varphi''(y)\varphi'(x) & & \end{pmatrix} \end{aligned}$$

whence $\varphi'(x)\varphi''(y) = \varphi(xy)$. Specializing to $x = 1$ or $y = 1$ we see $\varphi' = \varphi = \varphi''$. Thus $\varphi(x)\varphi(y) = \varphi(xy)$ which shows $\varphi \in \mathrm{Aut}(\mathbb{K})$. $\square$

Lemma 5.7. *Let $\tilde{A}_n^o := \{\mathrm{ad}(A) : A \in A_n^o\}$. Then $\tilde{A}_n^o(\mathbb{K}) \cap \mathfrak{aut}(\mathbb{K}) = \{0\}$.*

Proof. We have $\mathrm{ad}(A) \in \tilde{A}_n^o \cap \mathfrak{aut}(\mathbb{K}) \iff \mathrm{ad}(A)H_n(\mathbb{R}) = 0$ which implies $A = 0$: note that

$$\left[\begin{pmatrix} & -\bar{x} \\ x & \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right] = \begin{pmatrix} & 2\bar{x} \\ 2x & \end{pmatrix},$$

$$\left[\begin{pmatrix} a & \\ & b \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right] = \begin{pmatrix} & a-b \\ b-a & \end{pmatrix}.$$

Since $\mathrm{ad}(A)$ vanishes on real matrices, all off-diagonal entries of A vanish ($x = 0$) and all diagonal entries are equal ($a = b$). But trace $A = 0$, hence $A = 0$. □

Corollary 5.8. *Let $\mathfrak{f} \subset \mathfrak{aut}(H_3(\mathbb{O}))$ be the Lie subalgebra generated by $\tilde{A}_3^o$. Then $\mathfrak{f} = \tilde{A}_3^o \oplus \mathfrak{aut}(\mathbb{O})$.*

Proof. Since $\delta = [\mathrm{ad}(A), \mathrm{ad}(B)] - \mathrm{ad}([A,B]^o) \in \mathfrak{aut}(\mathbb{O}) \cap \mathfrak{f}$ for any $A, B \in A_3^o(\mathbb{O})$, we have $\mathfrak{f} \subset \hat{\mathfrak{f}} := \tilde{A}_3^o \oplus \mathfrak{aut}(\mathbb{O})$ (the sum is direct by Lemma 5.7). Further, $\mathrm{Aut}(\mathbb{O})$ acts on $\mathfrak{f}$ by conjugation: for any $A \in A_3^o(\mathbb{O})$ and $\phi \in \mathrm{Aut}(\mathbb{O})$ we have $\phi A \phi^{-1} = \phi(A) \in A_3^o(\mathbb{O})$ where $\phi(A)_{ij} = \phi(A_{ij})$, thus $\mathfrak{f}$ is an ideal in $\hat{\mathfrak{f}}$ containing $\tilde{A}_3^o$. Therefore $\mathfrak{f} \cap \mathfrak{aut}(\mathbb{O})$ is a nonzero ideal in $\mathfrak{aut}(\mathbb{O})$. But $\mathfrak{aut}(\mathbb{O})$ is simple, thus $\hat{\mathfrak{f}} \cap \mathfrak{aut}(\mathbb{O}) = \mathfrak{aut}(\mathbb{O})$ and hence $\mathfrak{f} = \hat{\mathfrak{f}}$. □

6. $\mathbb{OP}^2$ IS A SYMMETRIC SPACE

We show first that $\mathbb{OP}^2 \subset H_3(\mathbb{O})$ is an orbit of the group $F = \exp \mathfrak{f} \subset \mathrm{Aut}(H_3(\mathbb{O}))$. However, we do not even know yet that it is a submanifold. But consider the defining equation $P^2 = P$. If we let $P = \begin{pmatrix} \alpha & \bar{x} & \bar{y} \\ x & \beta & \bar{z} \\ y & z & \gamma \end{pmatrix}$, we have

$$\begin{array}{rcccl} x = P_{21} & = & (P^2)_{21} & = & (\alpha+\beta)x + \bar{z}y \\ y = P_{31} & = & (P^2)_{31} & = & (\alpha+\gamma)y + zx \\ z = P_{32} & = & (P^2)_{32} & = & (\beta+\gamma)z + y\bar{x} \end{array}$$

Thus zy is a multiple of x etc., hence x, y, z lie in a common quaternionic subalgebra $\tilde{\mathbb{H}} \subset \mathbb{O}$. Consequently, P is a rank-one projection matrix on $\tilde{\mathbb{H}}^3$. When $\tilde{\mathbb{H}} = \mathbb{H}$, these matrices form the orbit $\mathbb{HP}^2 \subset H_3(\mathbb{H})$ of $Sp_3 = \exp(A_3(\mathbb{H}))$. Since $A_3^o(\mathbb{H}) \subset A_3^o(\mathbb{O}) \subset \mathfrak{f}$ and $\mathbb{H}' \cdot I \subset \mathfrak{aut}(\mathbb{H}) \subset \mathfrak{aut}(\mathbb{O}) \subset \mathfrak{f}$, we have $A_3(\mathbb{H}) = A_3^o(\mathbb{H}) + \mathbb{H}' \cdot I \subset \mathfrak{f}$ and thus $Sp_3 \subset F$. Thus a subgroup of F acts transitively on $\mathbb{HP}^2$. Further, the group $G_2 = \mathrm{Aut}(\mathbb{O}) \subset F$ acts transitively on the space G_2/SO_4 of quaternionic subalgebras. Thus $\tilde{\mathbb{H}} = g\mathbb{H}$ for some $g \in \mathrm{Aut}\,\mathbb{O}$ and $\tilde{\mathbb{H}}\mathbb{P}^2 = g(\mathbb{HP}^2)$, hence F acts transitively on $\mathbb{OP}^2$.

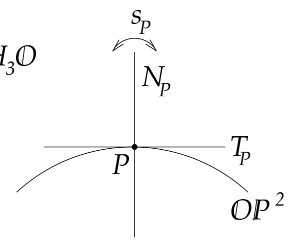
Now we can determine the tangent space T_P of $\mathbb{OP}^2 \subset H_3(\mathbb{O})$ for $P = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$. It is obtained by applying the full Lie algebra $\mathfrak{f}$ to P. However, we only know the subset $A_3^o \subset \mathfrak{f}$. For any $A = \begin{pmatrix} a & -v^* \\ v & A' \end{pmatrix} \in A_3^o$

with $v \in \mathbb{O}^2$ and $A' \in A_2(\mathbb{O})$ we have

$$[AP] = \begin{pmatrix} & v^* \\ v & \end{pmatrix} =: \hat{v}. \tag{6.1}$$

Thus $T_P \supset \hat{\mathbb{O}}^2 = \{\hat{v} : v \in \mathbb{O}^2\} \subset H_3(\mathbb{O})$. From the defining equation $P^2 = P$ we obtain the converse inclusion: Let $P(t)$ be a smooth family in $\mathbb{OP}^2$ with $P(0) = P = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$ and $P'(0) = X = \begin{pmatrix} \alpha & v^* \\ v & X' \end{pmatrix}$ with $v \in \mathbb{O}^2$ and $X' \in H_2(\mathbb{O})$. From $P^2 = P$ we obtain $P'P + PP' = P'$, hence $2P \circ X = X$. But $2P \circ X = 2 \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \circ \begin{pmatrix} \alpha & v^* \\ v & X' \end{pmatrix} = \begin{pmatrix} 2\alpha & v^* \\ v & 0 \end{pmatrix}$. Thus $2P \circ X = X$ implies $\alpha = 0$ and $X' = 0$, hence $X \in \hat{\mathbb{O}}^2$ and $T_P \subset \hat{\mathbb{O}}^2$, hence $T_P = \hat{\mathbb{O}}^2$ and consequently $N_P = \{\begin{pmatrix} \alpha & \\ & X' \end{pmatrix} : \alpha \in \mathbb{R},\ X' \in H_2(\mathbb{O})\}$.

Further we see that $\mathbb{OP}^2$ is a symmetric space, more precisely, it is *extrinsic symmetric*, that is: for any $P \in \mathbb{OP}^2$, the reflection s_P along the normal space N_P at P keeps $\mathbb{OP}^2$ invariant. Then s_P is called *extrinsic symmetry*.



In fact, the group SO_3 acts on $H_3(\mathbb{O})$ by conjugations $\mathrm{Ad}(U) : X \mapsto UXU^{-1}$ (with $U \in SO_3$ and $X \in H_3(\mathbb{O})$). Each $\mathrm{Ad}(U)$ is an automorphism on $H_3(\mathbb{O})$, thus it keeps $\mathbb{OP}^2$ invariant. In particular this holds for $\mathrm{Ad}(S)$ with $S = \begin{pmatrix} 1 & \\ & -I \end{pmatrix}$, and clearly N_P and T_P are the $+1$ and -1 eigenspaces of $\mathrm{Ad}(S)$. Thus $\mathrm{Ad}(S)$ is the extrinsic symmetry at P, and the extrinsic symmetry at an arbitrary point gP with $g \in F$ is $g\,\mathrm{Ad}(S)g^{-1}$. Since F acts transitively, $\mathbb{OP}^2$ is extrinsic symmetric. We will see in the next section that $\mathrm{Ad}(S)$ even belongs to $F \subset \mathrm{Aut}(H_3(\mathbb{O}))$.

7. The isotropy group of $\mathbb{OP}^2$ is $Spin_9$

We fix the element $P = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \in \mathbb{OP}^2 \subset H_3(\mathbb{O})$ (with $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$) as before. For $A = \begin{pmatrix} a & -v^* \\ v & A' \end{pmatrix} \in A_3^o$ with $v \in \mathbb{O}^2$ we have $[AP] = \begin{pmatrix} & v^* \\ v & \end{pmatrix} = \hat{v}$ (see (6.1)). This vanishes if and only if $v = 0$, hence $A = \begin{pmatrix} a & \\ & A' \end{pmatrix}$ with $A' = \begin{pmatrix} b & -\bar{x} \\ x & c \end{pmatrix}$ for $a, b, c \in \mathbb{O}'$ with $a + b + c = 0$ and $x \in \mathbb{O}$. As it will turn out later, it suffices to consider only the case $a = b = c = 0$, and for those A and all $v \in \mathbb{O}^2$ we have

$$[A\hat{v}] = \widehat{A'v} \text{ where } A' = \begin{pmatrix} & -\bar{x} \\ x & \end{pmatrix}. \tag{7.1}$$

We will see that the space of all such A' generates the Lie algebra of the group $Spin_9$ acting on $T_P \cong \mathbb{O}^2$ by the spin representation.

Recall that the *Clifford algebra* Cl_n is the associative algebra with 1 generated by euclidean n-space such that any orthogonal matrix on $\mathbb{R}^n$ extends naturally to an algebra isomorphism of Cl_n. The defining relations are

$$vw + wv = -2\langle v, w\rangle \cdot 1 \tag{7.2}$$

for all $v, w \in \mathbb{R}^n$, or equivalently,

$$e_i e_j + e_j e_i = -2\delta_{ij} \tag{7.3}$$

for any orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$. It contains euclidean $(n+1)$-space $\mathbb{R}^{n+1} = \mathbb{R} \cdot 1 \oplus \mathbb{R}^n$ whose unit sphere

$$\mathbb{S}^n = \{(\lambda, v) = \lambda \cdot 1 + v : \lambda^2 + |v|^2 = 1\}$$

consists of invertible elements: $(\lambda + v)(\lambda - v) = \lambda^2 + |v|^2 = 1$. They generate a subgroup of $Cl_n^\times$ (the group of invertible elements in Cl_n) which is the *spin group* $Spin_{n+1}$. The tangent space at 1 of $\mathbb{S}^n$ is $\mathbb{R}^n$; it is contained in the Lie algebra $T_1 Spin_{n+1} = \mathfrak{spin}_{n+1}$. Thus $\mathbb{R}^n$ generates $\mathfrak{spin}_{n+1}$ as a Lie algebra since $\mathbb{S}^n$ generates $Spin_{n+1}$ as a group: note that $\mathbb{S}^n = \exp(T_I \mathbb{S}^n)$ since for any $v \in \mathbb{R}^n$, $e^{tv} = \cos t + v \sin t$ is a one-parameter group in $Cl_n^\times$ and at the same time a great circle in $\mathbb{S}^n$. As a vector space, $\mathfrak{spin}_{n+1}$ is spanned by all e_i and $e_i e_j$ with $i < j$ where $e_1, \dots, e_n$ is an orthonormal basis of $\mathbb{R}^n$. Thus $\dim Spin_{n+1} = n + \frac{1}{2} n(n-1) = \frac{1}{2} n(n+1)$. In fact, for $j \neq k$ we have $e_j e_k = -e_k e_j$, hence $[e_j, e_k] = 2 e_j e_k$, and when i, j, k are distinct, $[e_i, [e_j, e_k]] = 2[e_i, e_j e_k] = 0$, while $[e_i [e_j, e_i]] = 2 e_i (e_j e_i - e_i e_j) = 4 e_j$.

The generating elements e_i and $e_{ij} := e_i e_j$ satisfy $e_{ij}^2 = -e_i^2 e_j^2 = -1$, thus $\exp(t e_{ij}) = \cos t + e_{ij} \sin t = 1 \iff t \in 2\pi\mathbb{Z}$. We will call them *unit generators* of $\mathfrak{spin}_{n+1}$.

Remark. Originally, $Spin_{n+1}$ was defined as a subset of Cl_{n+1}^+ (the space of even elements in Cl_{n+1}), and we have used the algebra isomorphism $Cl_n \to Cl_{n+1}^+$, $e_i \mapsto e_i e_{n+1}$ to embed it into Cl_n. Then the above basis elements $e_i, e_i e_j$ of $\mathfrak{spin}_{n+1}$ are replaced by $e_i e_{n+1}, e_i e_j$ (where $i, j \leq n$, $i < j$). This embedding shows better the connection to SO_{n+1}. Recall the subgroup $Pin_{n+1} \subset Cl_{n+1}^\times$ generated by $\mathbb{S}^n \subset \mathbb{R}^{n+1} \subset Cl_{n+1}$. It acts on $\mathbb{R}^{n+1}$ by conjugation $\mathrm{Ad}(v) : x \mapsto -vxv = vxv^{-1}$, $v \in \mathbb{S}^n$, $x \in \mathbb{R}^{n+1}$. We have $vxv = x$ when $x \perp v$ while $vxv = -x$ when $x \in \mathbb{R}v$, thus $-\mathrm{Ad}(v)$ is the reflection along the hyperplane $v^\perp$. Since the orthogonal group O_{n+1} is generated by hyperplane reflections, we have $\mathrm{Ad}(Pin_{n+1}) = O_{n+1}$. Now $Spin_{n+1}$ is the subgroup of Pin_{n+1} consisting of products with an even number of factors. Hence $\mathrm{Ad}(Spin_{n+1}) = SO_{n+1}$. This is a 2:1 covering with kernel $\{\pm 1\}$.[17]

A *representation* of Cl_n is an algebra homomorphism ϕ into some matrix algebra, $\phi : Cl_n \to \mathbb{R}^{p\times p}$, making the vector space $\mathbb{R}^p$ a Cl_n-module. By (7.3), a representation is given by any system of anti-commuting complex structures $J_1, \dots, J_n$ on $\mathbb{R}^p$, that means $J_i^2 = -I$ and $J_iJ_k = -J_kJ_i$ for $k \neq i$. There is essentially just one irreducible representation for each Cl_n; in particular, its dimension p is uniquely determined. Only if $n{+}1$ is a multiple of 4, there are two such representations: they differ by the automorphism of Cl_n induced by $-I$ on $\mathbb{R}^n$, and both are not faithful (only their direct sum is). When restricted to $Spin_{n+1} \subset Cl_n$, this representation is called *half spin representation* if $4|(n+1)$ and *spin representation* otherwise.

Example $n = 8$. The irreducible Cl_8-module is $\mathbb{O}^2 = \mathbb{R}^{16}$, and $\mathbb{O} = \mathbb{R}^8 \subset Cl_8$ acts on $\mathbb{O}^2$ as the space $\hat{\mathbb{O}}$ which we already have met in (7.1):

$$\hat{\mathbb{O}} = \{ \left(\begin{smallmatrix} & -\bar{x} \\ x & \end{smallmatrix} \right) : x \in \mathbb{O} \} .$$

This is a block matrix where the entries are left multiplications in $\mathbb{O}$, that is x stands for the (8×8)-matrix $L(x)$ which is the linear map $u \mapsto xu$ on $\mathbb{O}$. We check the Clifford relations (7.2):

$$\begin{pmatrix} & -L(\bar{x}) \\ L(x) & \end{pmatrix} \circ \begin{pmatrix} & -L(\bar{y}) \\ L(y) & \end{pmatrix} = -\langle x, y \rangle \cdot \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix} \right) .$$

Here we have used the octonionic identity

$$L(\bar{x})L(y) + L(\bar{y})L(x) = 2\langle x, y\rangle \cdot I = L(\bar{x}y + \bar{y}x). \tag{7.4}$$

In fact, when $x, y \in \mathbb{O}'$, (7.4) is the polarization of 2.2, page 6 which says $L(\bar{x})L(x) = -L(x)^2 = \langle x, x\rangle I$ for $x \in \mathbb{O}'$, and in the general case $x = \xi + x'$, $y = \eta + y'$ we have $L(\bar{x})L(y) + L(\bar{y})L(x) = 2\xi\eta + 2\langle x', y'\rangle = 2\langle x, y\rangle$.

By (7.1), the group $Spin_9 \subset Cl_8$ generated by $\hat{\mathbb{S}}^8 = \exp \hat{\mathbb{O}}$ is contained in the isotropy group of F at $P = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$. Counting dimensions we see that it is the full isotropy group (which is connected since $\mathbb{OP}^2$ is simply connected) since $\dim \mathfrak{f} - \dim \mathfrak{so}_9 = 52 - 36 = 16 = \dim \mathbb{OP}^2$. Further, the symmetry s_P corresponds to $-I \in Spin_9$, hence $s_P \in$

[17]Let $\alpha = v_1 \dots v_k \in Pin_{n+1}$ for some $k > 0$. Claim: When $\alpha e_1 = e_1\alpha$ then $v_1, \dots, v_r \perp e_1$. For this we write $\alpha = e_1\beta + \gamma$ where all terms of β and γ are products of vectors e_j, $j \geq 2$ of length $k - 1$ and k, respectively. Comparing αe_1 and $e_1\alpha$ we see $\beta = 0$ which shows the claim. The same conclusion holds for any e_j, thus $\alpha \in \mathbb{R}$ which shows $\alpha = \pm 1$ since it is a product of unit vectors.

$Spin_9 \subset F$. Therefore F contains all symmetries of $\mathbb{OP}^2$ and in particular it contains the identity component of its isometry group. Thus it is the identity component of $\mathrm{Aut}(H_3(\mathbb{O}))$ which acts isometrically and effectively on $\mathbb{OP}^2$.

We have seen that the isotropy group of $\mathbb{OP}^2$ is the group $Spin_9 \subset Cl_8 \cong \mathbb{R}^{16\times 16}$ which is generated by the unit sphere $\hat{\mathbb{S}}^8$ in

$$\hat{\mathbb{R}}^9 = \widehat{\mathbb{O}\oplus\mathbb{R}} = \left\{ \begin{pmatrix} \lambda & -\bar{x} \\ x & \lambda \end{pmatrix} : x \in \mathbb{O},\ \lambda \in \mathbb{R} \right\}, \tag{7.5}$$

Following the recent doctoral thesis of Erich Dorner [3] we will generalize (7.5) in a way which was first suggested by Boris Rosenfeld [15]. According to the classification of É. Cartan there are three exceptional symmetric spaces with dimensions 32, 64, 128, whose transvection groups[18] are the exceptional groups E_6, E_7, E_8. These are the so called *Rosenfeld planes*. Rosenfeld tried to describe them as projective planes over the algebra $\mathbb{O}\otimes\mathbb{L}$ with $\mathbb{L} = \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ (see also [2, page 313]. This did not work out properly, but still the isotropy group of these spaces can be described in terms of $\mathbb{O}\otimes\mathbb{L}$. We have to extend the coefficients λ in (7.5) from $\mathbb{R}$ to $\mathbb{L}$ and consider the space of matrices

$$\hat{\mathbb{R}}^{8+l} = \widehat{\mathbb{O}\oplus\mathbb{L}} = \left\{ \begin{pmatrix} \lambda & -\bar{x} \\ x & \bar{\lambda} \end{pmatrix} : x \in \mathbb{O},\ \lambda \in \mathbb{L} \right\} \tag{7.6}$$

which are matrices over the algebra $\mathbb{O}\otimes\mathbb{L} = \{\sum_{k=0}^{7} \lambda_k e_k : \lambda_k \in \mathbb{L}\}$ (where $\lambda_k = 1 \otimes \lambda_k$ and $e_k = e_k \otimes 1$). The unit sphere in $\widehat{\mathbb{O}\oplus\mathbb{L}}$ will generate the group $Spin_{8+l}$ where $l = \dim \mathbb{L}$. Again, the Clifford relations (7.2) are valid since the matrix $\left(\begin{smallmatrix} \lambda & \\ & -\lambda \end{smallmatrix}\right)$ for $\lambda \in \mathbb{L}'$ squares to $-|\lambda|^2 I$ and anticommutes with $\left(\begin{smallmatrix} & -\bar{x} \\ x & \end{smallmatrix}\right)$.

Unlike the octonionic plane, the Rosenfeld planes cannot be constructed easily as submanifolds of euclidean space. But we know its tangent space at one point together with its isotropy representation, and these data suffice to reconstruct the symmetric space. More precisely, the isotropy representation of a symmetric space $P = G/K$ determines P up to coverings. But of course there are much more representations than symmetric spaces, and only few of them, the so called *s-representations*, are isotropy representations of a symmetric space. We have to address the question why this representation of $Spin_{8+l}$ is an s-representation. In fact this is not even quite true: We

[18] The *transvection group* of a symmetric space P consists of compositions of an even number of symmetries. If P is compact, the transvection group is the identity component of the isometry group of P.

first have to enlarge this matrix group by right multiplications with those unit elements of $\mathbb{L}$ which commute with all left multiplications.

8. Representations of certain Clifford algebras

We generalize the representation of $Spin_9$ from the previous section to $Spin_{k+l}$ with $k,l \in \{1,2,4,8\}$ as follows. The Cl_{k+l-1} module is $(\mathbb{K}\otimes\mathbb{L})^2$ where $\mathbb{K}\otimes\mathbb{L}$ denotes the tensor product over $\mathbb{R}$ for arbitrary normed algebras $\mathbb{K}$ and $\mathbb{L}$. The representation is determined by the following embedding of $\mathbb{R}^{k+l} = \mathbb{K}\oplus\mathbb{L}$ into $(\mathbb{K}\otimes\mathbb{L})^{2\times 2}$, the space of 2×2-matrices with coefficients in $\mathbb{K}\otimes\mathbb{L}$, which can be viewed as real $(2kl\times 2kl)$-matrices:

$$\mathbb{K}\oplus\mathbb{L} \ni (x,\lambda) \mapsto \begin{pmatrix} \lambda & -\bar{x} \\ x & \bar{\lambda} \end{pmatrix} \tag{8.1}$$

where $x\in\mathbb{K} = \mathbb{K}\otimes 1$ and $\lambda\in\mathbb{L} = 1\otimes\mathbb{L}$. Thus the representation of $\mathbb{K}\oplus\mathbb{L}$ is spanned by the identity matrix I and $k+l-1$ anticommuting complex structures

$$\begin{pmatrix} \mu & \\ & -\mu \end{pmatrix}, \quad \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \quad \begin{pmatrix} & f \\ f & \end{pmatrix}, \tag{8.2}$$

where μ and f run through standard orthonormal bases of the imaginary parts of $\mathbb{K}$ and $\mathbb{L}$. The latter matrices span the tangent space of $\mathbb{S}^{n-1}\subset Spin_n$, and together with their Lie products they generate a matrix representation of the whole Lie algebra $\mathfrak{spin}_{k+l}$ which we call $\underline{\mathfrak{spin}}_{k+l}$. Commutators of these anticommuting matrices are twice their products which are again either diagonal or anti-diagonal. Thus the full Lie algebra $\mathfrak{spin}_{k+l}$ is spanned by the matrices in (8.2) together with the following ones:

$$\begin{pmatrix} & \mu \\ \mu & \end{pmatrix}, \begin{pmatrix} f & \\ & -f \end{pmatrix}, \begin{pmatrix} \mu\nu & \\ & \mu\nu \end{pmatrix}, \begin{pmatrix} ef & \\ & ef \end{pmatrix}, \begin{pmatrix} & \mu f \\ \mu f & \end{pmatrix}, \tag{8.3}$$

where ef means $L(e)L(f)$ (with $e<f$ with respect to the natural ordering of the orthonomal basis). We may assume that $\mathbb{L}\subset\mathbb{K}$ and that the chosen orthonormal basis of $\operatorname{Im}\mathbb{L}$ is contained in that of $\operatorname{Im}\mathbb{K}$.

Allowing now e,f,μ,ν to run through the full basis of $\mathbb{K}$ and $\mathbb{L}$, including the unit element 1, we can combine (8.2) and (8.3) to the following set of generators of $\underline{\mathfrak{spin}}_{k+l}$ as a vector space:

$$\begin{pmatrix} \mu\nu & \\ & \bar{\mu}\bar{\nu} \end{pmatrix}, \begin{pmatrix} ef & \\ & \bar{e}\bar{f} \end{pmatrix}, \begin{pmatrix} & -\bar{\mu}\bar{f} \\ \mu f & \end{pmatrix}. \tag{8.4}$$

Let $\underline{Spin}_{k+l}\subset\mathbb{R}^{2kl\times 2kl}$ be the represented group. As mentioned above, this is not yet the isotropy representation of a symmetric space, but we have to enlarge it by certain right multiplications: we put

$$S_{\mathbb{K}} = \begin{cases} \{R(x) : x \in \mathbb{K},\ |x| = 1\} & \text{for} \quad \mathbb{K} = \mathbb{C}, \mathbb{H} \\ \{I\} & \text{for} \quad \mathbb{K} = \mathbb{R}, \mathbb{O} \end{cases}$$

and we consider the matrix group

$$K = \underline{Spin}_{k+l} \cdot S_{\mathbb{K}} \cdot S'_{\mathbb{L}} \tag{8.5}$$

acting on $(\mathbb{K} \otimes \mathbb{L})^2$. Similarly we define $S'_{\mathbb{L}}$ with $R(x)$, $x \in \mathbb{K}$ replaced by $R(y')$, $y \in \mathbb{L}$. In the next two sections we will prove:

Theorem 8.1. *This representation of K is an s-representation, the isotropy representation of a symmetric space which we call the generalized Rosenfeld plane* $(\mathbb{K} \otimes \mathbb{L})\mathbb{P}^2$.

To understand the strategy of the proof we first have to recall how a simply connected symmetric space is reconstructed from its isotropy representation. The construction goes back to É. Cartan and was used by Ernst Witt [18] for the construction of the exceptional Lie group E_8.

In order to prove the theorem, we have to show that the vector space $\mathfrak{g} := \mathfrak{k} \oplus V$ with $V = (\mathbb{K} \otimes \mathbb{L})^2$ carries a Lie bracket extending the one on $\mathfrak{k}$, and $\mathfrak{k}$ and V are the (± 1)-eigenspaces of an *involution* on $\mathfrak{g}$, a Lie algebra automorphism σ with $\sigma^2 = \mathrm{id}$. The idea is simple: for all $A \in \mathfrak{k}$ and $v, w \in V$ we let $[A, v] = -[v, A] = Av$ and define $[v, w] \in \mathfrak{k}$ by its inner product with any elements in $\mathfrak{k}$:

$$[v, w] \in \mathfrak{k}, \quad \langle A, [v, w] \rangle_{\mathfrak{k}} := \langle Av, w \rangle \text{ for all } A \in \mathfrak{k}. \tag{8.6}$$

The inner product $\langle\, , \rangle_{\mathfrak{k}}$ on $\mathfrak{k}$ must be invariant under the adjoint action of K and such that at the end it is the restriction of an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. E.g. one can always choose the trace metric for the representation of K on $\mathfrak{g} = \mathfrak{k} \oplus V$, but there are other choices. If $\mathfrak{k}$ is simple, there is just one $\mathrm{Ad}(K)$-invariant metric on $\mathfrak{k}$ up to a scalar factor. In any case, this "Lie bracket" is invariant under K. Even more is true: it is invariant under the normalizer of K within the orthogonal group of V:

Lemma 8.2. *Let (V, K) be any representation, and V carries the Lie triple product . Let $\hat{K}$ be the normalizer of K in the orthogonal group of V. Then $\hat{K}$ consists of automorphisms of the "Lie triple" given by* (8.6)*, that means for all $k \in \hat{K}$ and $v, w \in V$,*

$$[kv, kw] = k[v, w]k^{-1}. \tag{8.7}$$

Proof. For all $A \in \mathfrak{k}$ we put $A' = k^{-1}Ak \in \mathfrak{k}$. then

$$\begin{aligned} \langle A, k[v,w]k^{-1}\rangle_{\mathfrak{k}} &= \langle A', [v,w]\rangle \\ &\overset{(8.6)}{=} \langle A'v, w\rangle \\ &= \langle Akv, kw\rangle \\ &\overset{(8.6)}{=} \langle A, [kv,kw]\rangle_{\mathfrak{k}} \end{aligned}$$

□

So far, this construction is quite general. It depends only on a representation (K, V) of any compact Lie group K and an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}$. The decisive point to check is the Jacobi identity (5.5) for the new "Lie bracket" (8.6) which fails for most representations. However, by (8.7) it is enough to prove (5.5) on V,

$$[v,w]u + [w,u]v + [u,v]w = 0 \text{ for all } u,v,w \in V. \tag{8.8}$$

In our case this will follow easily using subrepresentations which are already known to be s-representations, see next section. When (8.8) holds, V is called a *Lie triple.*

9. $(\mathbb{K} \otimes \mathbb{L})^2$ is always a Lie triple

Let us first consider the largest associative algebra of type $\mathbb{K}\otimes\mathbb{L}$, that is $\mathbb{H} \otimes \mathbb{H}$. This occurs as a subalgebra of the largest case $\mathbb{O} \otimes \mathbb{O}$ which is the main observation for the proof of Theorem 8.1, see Theorem 9.4 below. Now the two tensor factors will play a symmetric rôle, so we cannot stay with our convention denoting the first factor by latin and the second one by greek letters. Instead we will put $x = x \otimes 1$ and $x' = 1 \otimes x$ for each $x \in \mathbb{H}$, and we will stop using the notation x' for the imaginary part of x.

Lemma 9.1. *The representation of* $K = \underline{Spin}_8 \cdot S_{\mathbb{H}} \cdot S'_{\mathbb{H}}$ *on the vector space* $(\mathbb{H} \otimes \mathbb{H})^2$ *is isomorphic to the isotropy representation of the Grassmannian* $\mathsf{G}_4(\mathbb{R}^{12})$.

Proof. We consider the two involutions $R(ii')$ and $R(jj')$ in $SS' := S_{\mathbb{H}} \cdot S'_{\mathbb{H}} = (\mathbb{S}^3 \times \mathbb{S}^3)/\pm$. Since they commute with each other, they have a common eigenspace decomposition

$$V = E_{++} + E_{+-} + E_{-+} + E_{--}$$

where E_{++} is the intersection of the $(+1)$-eigenspaces of $R(ii')$ and $R(jj')$ etc. On the other hand, $R(j), R(k)$ and $R(i), R(k)$ anticommute with $R(ii')$ and $R(jj')$ respectively, thus they change the signs of the eigenvalues and permute the eigenspaces. So all eigenspaces have the same dimension $\frac{1}{4} \cdot 32 = 8$. The matrix group $\underline{Spin}_8 \subset K$ commutes with all right translations, thus it keeps these four subspaces invariant. We put $E = E_{++}$ and consider the linear map

$$F : E \otimes \mathbb{H} \to (\mathbb{H} \otimes \mathbb{H})^2 : v \otimes p \mapsto vp = v\bar{p}' \tag{9.1}$$

(recall that $R(p) = -R(p')$ on E for all imaginary $p \in \mathbb{H}$). The action of $Spin_8$ lives only on $E \cong \mathbb{R}^8$; it is a nontrivial homomorphism $Spin_8 \to SO_8$ which must be surjective since $\mathfrak{spin}_8 = \mathfrak{so}_8$ is a simple Lie algebra. On the other hand, the action of SS' transforms only the scalar p in (9.1): for any $a, b \in S_{\mathbb{H}} = \mathbb{S}^3$ and $v \in E$, the action of (a, b) on vp is by

$$a.vp := vp\bar{a}, \quad b.vp := vp\bar{b}' = v\bar{b}'p = vbp.$$

In other words, $\pm(a, b) \in SS'$ acts only on the second tensor factor of $E \otimes \mathbb{H}$, and it acts by $p \mapsto bp\bar{a}$, which is the standard action of

$$SO_4 = \{L(b)R(\bar{a}) : b, a \in \mathbb{S}^3\}$$

on $\mathbb{H} = \mathbb{R}^4$. The linear map F in (9.1) is a linear isomorphism (onto and the same dimensions 32 for domain and range) which is K-equivariant with respect to the representation of $K = \underline{Spin}_8 \cdot SS'$ as tensor representation of $SO_8 \times SO_4$ on $E \otimes \mathbb{H}$. But this is the isotropy representation of the Grassmannian $\mathsf{G}_4(\mathbb{R}^{12})$. $\square$

Lemma 9.2. *K is contained in $\underline{Spin}_{16}$, and the inner products on $\mathfrak{k} \cong \mathfrak{spin}_8 \oplus \mathfrak{spin}_4$ induced from* Ad*-invariant inner products on $\mathfrak{spin}_{16}$ and on $\mathfrak{spin}_{12}$ are proportional.*

Proof. Clearly $\underline{Spin}_{4+4} \subset \underline{Spin}_{8+8}$. Left translations with unit basis elements acting on $\mathbb{O} \otimes \mathbb{O}$ have squared norm $\dim(\mathbb{O} \otimes \mathbb{O}) = 64$, thus all the generators of the Lie algebra in (8.4) have the same squared norm 128 in the trace metric of $\underline{Spin}_{16}$.

We have to show first that $SS' \subset \underline{Spin}_{16}$. We observe that for any $q \in \mathbb{H}$, the right translation $R(q)$ on $\mathbb{H}$ can be replaced by a composition of octonionic left translations:

$$R(q) = -L(l)L(\bar{q}l) \tag{9.2}$$

where $\mathbb{O} = \mathbb{H} \oplus l\,\mathbb{H}$ with $l^2 = -1$. In fact, if $q \in \mathbb{R}$, this holds trivially. If $q \perp \mathbb{R}$ (that is $\bar{q} = -q$) and $|q| = 1$, then $l \perp lq$, and for any $x \in \mathbb{H}$ we have

$$L(l)L(ql)x = l((ql)x) = \left\{\begin{array}{cccc} -l^2q & = & q & \text{if} \quad x = 1 \\ -l^2q^2 & = & -1 & \text{if} \quad x = q \\ l^2qx & = & -qx & \text{if} \quad x \perp 1, q \end{array}\right\} = xq$$

where we have used anti-associativity $l((lq)x) = -(l(lq))x$ in the case $x \perp 1, q$. A similar statement holds for l' and q' instead of l and q. By (8.3), the matrices $ef \cdot \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ and $e'f' \cdot \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)$ with $e = -l$ and $f = lq$ are contained in $\underline{\mathfrak{spin}}_{16}$ and hence[19] the group SS' of unit $(\mathbb{H}\otimes\mathbb{H})$-scalar multiplications is contained in $\underline{Spin}_{16}$.

The three components of $\mathfrak{k} \cong \mathfrak{spin}_8 \oplus \mathfrak{spin}_3 \oplus \mathfrak{spin}_3$ are perpendicular with respect to every $\mathrm{Ad}(K)$-invariant inner product since they are inequivalent submodules for the adjoint representation. The unit generators of the Lie algebra of SS' are matrices of type $\left(\begin{smallmatrix} L & \\ & L \end{smallmatrix}\right)$ with $L = L(e)L(f)$ or $L(e')L(f')$ where $e, f \in \mathrm{Im}\,\mathbb{O}$ are certain elements of the standard basis $B_{\mathbb{O}}$. Every left multiplication with an element of $B_{\mathbb{O}}$ is a permutation of $B_{\mathbb{O}}$ up to signs, hence the same holds for L, and its squared norm is the dimension 64 of $\mathbb{O}\otimes\mathbb{O}$. Thus in $\underline{\mathfrak{spin}}_{16}$, the unit generators of the Lie algebra of SS' have the same squared norm 128 as the unit generators of $\underline{\mathfrak{spin}}_8$.

Further, the standard basis elements e_ie_j, $1 \le i < j \le 12$ of $\mathfrak{spin}_{12} \subset Cl^+_{12}$ have all the same length. In particular this holds for those e_ie_j with $i, j \le 8$ or $i, j \ge 9$ which are the unit generators of $\mathfrak{spin}_8$ and $\mathfrak{spin}_4$. Thus the metrics on $\mathfrak{k}$ induced by $\mathfrak{spin}_{16}$ and $\mathfrak{spin}_{12}$ are proportional. $\square$

Corollary 9.3. *Let $\mathbb{H}_1, \mathbb{H}_2 \subset \mathbb{O}$ two subalgebras isomorphic to $\mathbb{H}$. Then (8.8) holds on the subtriple $(\mathbb{H}_1\otimes\mathbb{H}_2)^2 \subset (\mathbb{O}\otimes\mathbb{O})^2$.*

Proof. By the preceding two lemmas, the "Lie triple" (8.6) restricted to $(\mathbb{H}_1\otimes\mathbb{H}_2)^2$ is proportional to the Lie triple of the Grassmannian $\mathsf{G}_4(\mathbb{R}^{12})$ where (8.8) holds. $\square$

Theorem 9.4. *For any $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, the action of the group $K = \underline{Spin}_{k+l} \cdot S_{\mathbb{K}} \cdot S'_{\mathbb{L}}$ on $V = (\mathbb{K}\otimes\mathbb{L})^2$ is an s-representation. The Lie triple product on V is given by (8.6) for the metric on $\mathfrak{k} \subset \underline{\mathfrak{spin}}_{16}$ which is induced from the trace inner product on $\underline{\mathfrak{spin}}_{16}$.*

Proof. Since we always take the same inner product induced from the trace on $\underline{\mathfrak{spin}}_{16}$, we have to check (8.8) only for $(\mathbb{O}\otimes\mathbb{O})^2$, then it holds also for $(\mathbb{K}\otimes\mathbb{L})^2 \subset (\mathbb{O}\otimes\mathbb{O})^2$. It is enough to check (8.8) when a, b, c are basis elements. We consider the orthonormal basis

$$\hat{B} = B\mathbf{e}_1 \cup B\mathbf{e}_2 \tag{9.3}$$

of $V = (\mathbb{O}\otimes\mathbb{O})^2$ where $\mathbf{e}_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $\mathbf{e}_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and where B is the basis of $\mathbb{O}\otimes\mathbb{O}$ given by the tensor products of the standard basis of $\mathbb{O}$

$$B_{\mathbb{O}} = \{1, i, j, k, l, il, jl, kl\} \tag{9.4}$$

[19] If a matrix Lie algebra contains a complex structure J (that is $J^2 = -I$), then J is also contained in the corresponding matrix group since $e^{(\pi/2)J} = J$, like $e^{(\pi/2)i} = i$.

in each tensor factor. Using the equivariance for $K = \underline{Spin}_{16}$ (see (8.7)), we may assume that $c = \mathbf{e}_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. In fact, after applying one of the anti-diagonal matrices in (8.4) if necessary (all matrices in (8.4) lie in the intersection $\underline{Spin}_{16} \cap \underline{\mathfrak{spin}}_{16}$ since the commutator of these matrices is twice their product), we may assume that $c = \gamma\mathbf{e}_1$ for some $\gamma = e \otimes f = ef'$. Applying the inverse of $\mathrm{diag}(e, -e) \cdot \mathrm{diag}(f', -f')$ we change c to $\mathbf{e}_1$, and $\pm\hat{B}$ is kept invariant under this transformation.

The remaining basis elements are $a = \alpha\mathbf{e}_i$ and $b = \beta\mathbf{e}_j$ (where $i, j \in \{1, 2\}$) with $\alpha = uv'$ and $\beta = wx'$ in B. But the two octonions u, w lie in a common quaternionic subalgebra $\mathbb{H}_1 \subset \mathbb{O}$, and similarly $v, x \in \mathbb{H}_2$. Thus we are in a "classical" subspace of the form $(\mathbb{H}_1 \otimes \mathbb{H}_2)^2 \subset (\mathbb{O} \otimes \mathbb{O})^2$ where the Jacobi identity already holds, see the previous corollary. Thus we have established a Lie triple structure on $V = (\mathbb{K} \otimes \mathbb{L})^2$ and Lie algebra structure on $\mathfrak{g} = \mathfrak{k} \oplus V$ in all cases. $\square$

10. Rosenfeld lines

First we restrict our representation of K on $V = (\mathbb{K} \otimes \mathbb{L})^2$ to the subspace $V_1 = (\mathbb{K} \otimes \mathbb{L})^1$ which consists of the vectors $\left(\begin{smallmatrix}x\\y\end{smallmatrix}\right) \in V$ with $y = 0$. This is the fixed space of the reflection $r = \left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)$ on $(\mathbb{K} \otimes \mathbb{L})^2$ which normalizes K: it commutes with the diagonal matrices in (8.4) while conjugation with the antidiagonal matrices by r changes sign. By Lemma 8.2, r is an automorphism of the Lie triple, hence V_1 is a Lie subtriple, and the corresponding symmetric subspace of the generalized Rosenfeld plane $(\mathbb{K} \otimes \mathbb{L})\mathbb{P}^2$ is called (generalized) *Rosenfeld line* $(\mathbb{K} \otimes \mathbb{L})\mathbb{P}^1$.

Theorem 10.1. *Up to coverings, the Rosenfeld line $(\mathbb{K} \otimes \mathbb{L})\mathbb{P}^1$ is the Grassmannian $\mathsf{G}_k(\mathbb{R}^{k+l})$ where $k = \dim \mathbb{K}$ and $l = \dim \mathbb{L}$.*

Proof. The subgroup $K_1 \subset K$ preserving this subspace consists of the diagonal matrices in K, and only the upper left entry matters. According to (8.4), the Lie algebra for this representation is spanned by $L(e), L(f'), L(ef), L(e'f')$, and $R(ef')$ (due to the factor $S_{\mathbb{K}} \cdot S'_{\mathbb{L}}$, see (8.5)). Thus K_1 acts on $\mathbb{K} \otimes \mathbb{L}$ by left and right multiplications with several elements of $\mathbb{K} \otimes 1$ and $1 \otimes \mathbb{L}$. This is a tensor representation: the left and right multiplications by $\mathbb{K} = \mathbb{K} \otimes 1$ act only on the first tensor factor $\mathbb{K}$ while those by $\mathbb{L}' = 1 \otimes \mathbb{L}$ act only on the second tensor factor $\mathbb{L}$. The infinitesimal action of K_1 on the tensor factor $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ is generated by the action of $\mathrm{Im}\,(\mathbb{K})$ through right multiplication on $\mathbb{K}$. This is the natural action of SO_1, SO_2, SO_4, SO_8, respectively. The same holds for $\mathbb{L}$. Thus the image of the representation of K_1 on $\mathbb{K} \otimes \mathbb{L}$ is the tensor product action of $SO_k \times SO_l$ on $\mathbb{K} \otimes \mathbb{L}$, which is the isotropy representation of the Grassmannian $\mathsf{G}_k(\mathbb{R}^{k+l})$ of k-dimensional linear subspaces in $\mathbb{R}^{k+l}$.[20]

11. The Lie triple on $(\mathbb{K}\otimes\mathbb{L})^2$

The Lie triple can be computed by (8.6): For any $v,w \in V = (\mathbb{K}\otimes\mathbb{L})^2$ we have $[v,w] \in \mathfrak{k}$ with

$$\langle A,[v,w]\rangle_{\mathfrak{k}} = \langle Av, w\rangle \tag{11.1}$$

for all $A \in \underline{\mathfrak{spin}}_{k+l}$, see (8.4). As inner product on $\mathfrak{k}$ we choose the one induced by the trace inner product on $\mathfrak{spin}_{16}$. Using the orthonormal basis $\mathfrak{B}$ of $\underline{\mathfrak{spin}}_{16}$ as given in (8.4) we see $[v,w] = \sum_j \langle A_j v, w\rangle A_j$ and

$$[u,[v,w]] = -[v,w]u = -\sum_j \langle A_j v, w\rangle A_j u \tag{11.2}$$

where $A_j \in \mathfrak{B}$ with $\langle A_j v, w\rangle \neq 0$. Since we want that $(\mathbb{K}\otimes\mathbb{L})^2$ is a Lie subtriple of $(\mathbb{O}\otimes\mathbb{O})^2$, we may compute the Lie triple product $[u,[v,w]]$ for all $u,v,w \in (\mathbb{K}\otimes\mathbb{L})^2$ within $\mathbb{O}\otimes\mathbb{O}$. The base $\mathfrak{B}$ of $\underline{\mathfrak{spin}}_{16}$ acts on $\pm\hat{B}$ of $(\mathbb{O}\otimes\mathbb{O})^2$, see (9.3): any $A \in \mathfrak{B}$ permutes $\hat{B}$ up to sign.

1. Let us compute $[\mathbf{e}_1,\mathbf{e}_2]$. We have to look for $A \in \mathfrak{B}$ with $\langle A,[\mathbf{e}_1,\mathbf{e}_2]\rangle = \langle A\mathbf{e}_1,\mathbf{e}_2\rangle \neq 0$. The only $A \in \mathfrak{B}$ carrying $\mathbf{e}_1$ to a real multiple of $\mathbf{e}_2$ is $A = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$ with $A\mathbf{e}_1 = \mathbf{e}_2$, thus $[\mathbf{e}_1,\mathbf{e}_2] = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$. Further,

$$[\mathbf{e}_1,[\mathbf{e}_1,\mathbf{e}_2]] = -A\mathbf{e}_1 = -\mathbf{e}_2.$$

More generally, we have:

Lemma 11.1. *Let B be the canonical orthonormal basis of $\mathbb{K}\otimes\mathbb{L}$. Then for any $\alpha,\beta \in B$ we have*

$$[\alpha\mathbf{e}_1,\beta\mathbf{e}_2] = \begin{pmatrix} & -\alpha\bar\beta \\ \beta\bar\alpha & \end{pmatrix} \neq 0. \tag{11.3}$$

Proof. According to (8.4), we have $\langle A,[\alpha\mathbf{e}_1,\beta\mathbf{e}_2]\rangle = \langle A\alpha\mathbf{e}_1,\beta\mathbf{e}_2\rangle \neq 0$ $\iff$ $A = A_\gamma := \left(\begin{smallmatrix} & -\bar\gamma \\ \gamma & \end{smallmatrix}\right)$ with $\gamma\alpha = \beta$, hence $\gamma = \beta\bar\alpha$. $\square$

[20]Recall that the k-planes in $\mathbb{R}^n$ close to $\mathbb{R}^k = \{x \in \mathbb{R}^n : x_{k+1} = \dots = x_n = 0\}$ are just the graphs of linear maps $F : \mathbb{R}^k \to (\mathbb{R}^k)^\perp = \mathbb{R}^{n-k} = \mathbb{R}^l$. Therefore the tangent space of $\mathsf{G}_k(\mathbb{R}^{k+l})$ at $\mathbb{R}^k$ is $\mathrm{Hom}(\mathbb{R}^k,\mathbb{R}^l)$. The group SO_n acts transitively on $\mathsf{G}_k(\mathbb{R}^n)$, and the isotropy group of the (oriented) k-plane $\mathbb{R}^k \in \mathsf{G}_k(\mathbb{R}^n)$ is $SO_k \times SO_l$ acting on the tangent space $\mathrm{Hom}(\mathbb{R}^k,\mathbb{R}^l)$ by $((A,B),F) \mapsto BFA^{-1}$. Using the identification $\mathrm{Hom}(\mathbb{R}^k,\mathbb{R}^l) = \mathbb{R}^k\otimes\mathbb{R}^l$, this is the tensor representation

$$((A,B), v\otimes w) \mapsto Av\otimes Bw.$$

2. Next we compute $[\mathbf{e}_1, i\mathbf{e}_1]$. This is tangent to the Rosenfeld line whose structure as a symmetric space we already know, but we still have to see the induced metric. In most cases, the metric is unique up to a factor (by irreducibility), but it remains to determine this factor. Again we have to look for all $A \in \mathfrak{B}$ with $\langle Ae_1, ie_1\rangle \neq 0$. This time, there are four such matrices A, and they satisfy $A\mathbf{e}_1 = i\mathbf{e}_1$: these are $\left(\begin{smallmatrix} i & \\ & -i \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} L_s & \\ & L_s \end{smallmatrix}\right)$, $s = 1, 2, 3$, where $L_1 = L(j)L(k)$ and $L_2 = L(l)L(il)$ and $L_3 = L(kl)L(jl)$. In fact, there are four possibilities to represent i as a product of two octonions in $B_{\mathbb{O}}$:

$$i = 1 \cdot i = jk = l(il) = (kl)(jl).$$

The three representations by imaginary octonions correspond to the three triangles through each vertex in our heptagon, see figure on page 10. Thus $[\mathbf{e}_1, i\mathbf{e}_1]$ has a component in each of these four basis matrices $A_1, \dots, A_4$, it is the sum of these four matrices:

$$[e_1, ie_1] = \sum_j \langle A_j, [e_1, ie_1]\rangle A_j = \sum_j \langle A_j e_1, ie_1\rangle A_j = \sum_j A_j$$

where the metric is normalized so that the basis elements (8.4) have unit length (trace metric divided by 128). Since $A_j e_1 = ie_1$ for all four j, we obtain

$$[\mathbf{e}_1, [\mathbf{e}_1, i\mathbf{e}_1]] = -4i\mathbf{e}_1.$$

Thus the operator $-\operatorname{ad}(\mathbf{e}_1)^2$ measuring the curvature has eigenvalues 1 and 4 as we know already in $\mathbb{CP}^2$.

12. Maximal abelian subspace and roots

Changing our previous convention we now assume $\mathbb{K} \subset \mathbb{L}$. The Lie triple of the Grassmannian $\mathsf{G}_k(\mathbb{R}^{k+l})$ with $l \geq k$ is $V_1 = \mathbb{R}^{k \times l} = \mathbb{R}^k \otimes \mathbb{R}^l$, and its maximal abelian subspace is the set $\Sigma \subset V_1$ of matrices $(D, 0)$ where $D = \operatorname{diag}(t_1, \dots, t_k)$ is any real diagonal matrix. Thus a basis of Σ is given by $\{e_i \otimes e_i : i = 1, \dots, k\}$. In our representation as Rosenfeld lines we have $V_1 = \mathbb{K} \otimes \mathbb{L}$, and a basis of Σ is given by $e \otimes e = ee'$ where e runs through the standard basis $B_{\mathbb{K}} = B_{\mathbb{O}} \cap \mathbb{K}$ of $\mathbb{K}$. In the algebra $\mathbb{K} \otimes \mathbb{L}$, the subset Σ is a commutative and associative subalgebra.

Now we observe that Σ is also maximal abelian for V where $\mathbb{K} \otimes \mathbb{L}$ is viewed as a subspace of $V = (\mathbb{K} \otimes \mathbb{L})^2$ in the natural way. This is a consequence of Lemma 11.1. In fact, for $0 \neq \delta = \sum t_i \beta_i$ with $\beta_i \in B$ and $t_i \in \mathbb{R}$ we have

$$[\mathbf{e}_1, \delta\mathbf{e}_2] = \sum_i t_i \begin{pmatrix} & -\bar\beta_i \\ \beta_i & \end{pmatrix} = \begin{pmatrix} & -\bar\delta \\ \delta & \end{pmatrix}.$$

Thus $\mathbf{e}_1 \in \Sigma$ does not commute with $\delta\mathbf{e}_2$ which shows that the abelian subspace Σ cannot be enlarged.

Now we want to compute the common eigenspace decomposition on V for $-\operatorname{ad}(\gamma)^2$ for all $\gamma \in \Sigma$ (not just in $\Sigma \cap B$). The square roots of those values are the *roots* of the corresponding symmetric space, see [9]. The eigenspaces in $V_1 = \mathbb{KL}\mathbf{e}_1$ are already known from the Rosenfeld lines. It remains to compute $-\operatorname{ad}(\alpha)\operatorname{ad}(\beta)$ on $V_2 = \mathbb{KL}\mathbf{e}_2$ for any $\alpha, \beta \in \Sigma \cap B$. For all $\omega \in \mathbb{KL}$ we have

$$-[\alpha\mathbf{e}_1, [\beta\mathbf{e}_1, \omega\mathbf{e}_2]] = -[\alpha\mathbf{e}_1, A_{\omega\beta}] = (\omega\beta)\alpha\mathbf{e}_2 = R(\alpha)R(\beta)\omega\mathbf{e}_2$$

(Note that $\bar\beta = \beta$ when $\beta \in \Sigma \cap B = \{ee' : e \in B_{\mathbb{K}}\}$ where $B_{\mathbb{K}} = \mathbb{K} \cap B_{\mathbb{O}}$ is the standard basis of $\mathbb{K}$.) Consequently:

Lemma 12.1. *For all $\gamma \in \Sigma$ we have*

$$-\operatorname{ad}(\gamma\mathbf{e}_1)^2\omega\mathbf{e}_2 = (R(\gamma)^2\omega)\mathbf{e}_2. \tag{12.1}$$

Hence the roots on V_2 are the common eigenvalues of $R(\gamma)$, $\gamma \in \Sigma$.

Thus we just need to find the common eigenspaces of $R(\Sigma)$ on $\mathbb{KL}$. First we look for the eigenspaces inside Σ. We are using the standard bases $B_{\mathbb{O}} = \{1, i, j, ij, l, il, jl, (ij)l\}$ for $\mathbb{O}$ and $B_{\mathbb{K}} = B_{\mathbb{O}} \cap \mathbb{K}$ for $\mathbb{K} \subset \mathbb{O}$. In the case $\mathbb{K} = \mathbb{O}$ we choose $s_i, s_j, s_l \in \{\pm 1\}$ arbitrary, obtaining eight elements $\nu \in \Sigma$,

$$\begin{aligned} \nu &= (1 + s_i\hat{i})(1 + s_j\hat{j})(1 + s_l\hat{l}) \qquad (12.2)\\ &= 1 + s_i\hat{i} + s_j\hat{j} + s_is_j\,\hat{i}\hat{j} + s_l\hat{l} + s_is_l\,\hat{i}\hat{l} + s_js_l\,\hat{j}\hat{l} + s_is_js_l\,\hat{i}\hat{j}\hat{l}\\ &= \sum_{e \in B_{\mathbb{O}}} s_e\hat{e} \end{aligned}$$

where $\hat{i} := ii'$ etc. and where the other s_e are multiplicative: $s_1 = 1$, $s_{ij} = s_is_j$, $s_{il} = s_is_l$, $s_{jl} = s_js_l$, $s_{ijl} = s_is_js_l$. From (12.2) we see: multiplying ν by $\hat{i}$, $\hat{j}$, or $\hat{l}$ gives ν back, multiplied by s_i, s_j, s_l, respectively (remind that Σ is commutative and $\hat{e}^2 = 1$ for any $\hat{e} \in \Sigma \cap B$). By the multiplicativity of the s_e the same is true for arbitrary basis elements: $\nu\hat{e} = s_e\nu$. Thus for any $\tau = \sum t_e\hat{e} \in \Sigma$ we have

$$R(\tau)\nu = \lambda(\tau)\nu, \quad \lambda(\tau) = \sum s_et_e. \tag{12.3}$$

Thus we obtain 8 perpendicular common eigenvectors $\nu \in \Sigma$ for $R(\tau)$, $\tau \in \Sigma$. The corresponding eigenvalues ("roots") are the linear forms $\hat{e} \mapsto s_e$ on Σ. Note that the number of negative s_e is zero or four.

Similar, in the case $\mathbb{K} = \mathbb{H}$ we use the vectors

$$\nu = (1 + s_i\hat{i})(1 + s_j\hat{j}) = 1 + s_i\hat{i} + s_j\hat{j} + s_k\hat{k}$$

with arbitrary $s_i, s_j \in \{\pm 1\}$ and $s_k = s_i s_j$, which decompose Σ into 4 common eigenspaces for $R(\hat{e})$, $e \in B_{\mathbb{H}}$, and the number of negative s_e is zero or two. Likewise for $\mathbb{K} = \mathbb{C}$ we choose

$$\nu = 1 + s_i \hat{i}$$

for $s_i = \pm 1$. These are eigenvectors of $\hat{i}$ forming a basis of Σ, and the eigenvalue is $s_i = \pm 1$. If $\mathbb{K} = \mathbb{R}$, then $\Sigma = \mathbb{R} \cdot 1$ and the only eigenvector (with multiplicity 1) is $\nu = 1$. We omit this case in the following.

Next we observe[21] that $R(f')\nu$ for any $f' \in B_{\mathbb{L}} \setminus \{1\}$ is still a common eigenvector for $R(\hat{e})$, $e \in B_{\mathbb{K}}$. In fact, $R(f')$ commutes or anticommutes with $R(\hat{e})$ since

$$R(ee')R(f') = R(e')R(f')R(e) = \begin{cases} +R(f')R(ee') & \text{if } e \in \{1, f\}, \\ -R(f')R(ee') & \text{else.} \end{cases}$$

Thus $R(\hat{e})R(f')\nu = \pm s_e R(f')\nu$ with "+" for $e \in \{1, f\}$ and "$-$" otherwise. The number of orthonormal eigenvectors we obtain in this way is $k \cdot l = \dim V_2$, hence we have got a basis of V_2 by common eigenvectors for $R(\hat{e})$, $e \in B_{\mathbb{K}}$. The corresponding roots are of the form

$$\lambda = \sum_{e \in B_{\mathbb{K}}} s_e \phi_e \tag{12.4}$$

where $(\phi_e)_{e \in B_{\mathbb{O}}}$ is the basis of Σ^* dual to $\hat{B}_{\mathbb{K}} = \{ee' : e \in B_{\mathbb{K}}\} = \Sigma \cap B$, and $s_e \in \{\pm 1\}$. The eigenvalues s_e of $R(\hat{e})$ corresponding to the eigenvectors ν and $R(f')\nu$ are the same for $\hat{e} = 1$ and $\hat{e} = \hat{f}$ and different for the other $k - 2$ basis elements $\hat{e} \in B_{\mathbb{K}}$.

There are 2^k linear forms $\sum_{e \in B_{\mathbb{K}}} s_e \phi_e$ with $s_e = \pm 1$ (now without any multiplicativity). Since only λ^2 matters, we may assume $s_1 = 1$ by changing all s_e to $-s_e$ when $s_1 = -1$. Moreover, in the cases $\mathbb{O}\mathbb{O}$ and $\mathbb{H}\mathbb{H}$ the number of negative s_e is even, that is $\prod_e s_e = 1$, which reduces the number N of possible linear forms to 2^{k-2}, that is to 64 for $\mathbb{O}\mathbb{O}$ and to 4 for $\mathbb{H}\mathbb{H}$. In the other cases $\mathbb{H}\mathbb{O}$ and $\mathbb{C}\mathbb{L}$ we obtain 2^{k-1} such forms, that is 8 and 2, respectively. Thus all possible linear forms actually occur as roots, and their multiplicities are $\dim \mathbb{K}\mathbb{L}/N$ which is $64/64 = 1$ for $\mathbb{O}\mathbb{O}$, $32/8 = 4$ for $\mathbb{H}\mathbb{O}$, $16/4 = 4$ for $\mathbb{H}\mathbb{H}$, $16/2 = 8$ for $\mathbb{C}\mathbb{O}$, $8/2 = 4$ for $\mathbb{C}\mathbb{H}$ and $4/2 = 2$ for $\mathbb{C}\mathbb{C}$.

In Table 1 below, we show the root system R_1 and R on $V_1 = \mathbb{K} \otimes \mathbb{L}$ and $V = (\mathbb{K} \otimes \mathbb{L})^2$ with multiplicities m,

(a) first the roots on V_1 for the Rosenfeld line,

(b) then the remaining roots on V_2 for the Rosenfeld plane.

[21] For $f \in \mathbb{K}$ we have $R(f)\nu = \mp R(f')\nu$ since $R(ff')\nu = \pm \nu$.

No	$\mathbb{K}\otimes\mathbb{L}$	k	(a) λ	(a) m	(a) R_1	(b) λ	(b) m	(b) R
1	$\mathbb{R}\otimes\mathbb{R}$	1	none	0	$\emptyset$	ϕ_1	1	A_1
2	$\mathbb{R}\otimes\mathbb{C}$	1	$2\phi_1$	1	A_1	ϕ_1	2	BC_1
3	$\mathbb{R}\otimes\mathbb{H}$	1	$2\phi_1$	3	A_1	ϕ_1	4	BC_1
4	$\mathbb{R}\otimes\mathbb{O}$	1	$2\phi_1$	7	A_1	ϕ_1	8	BC_1
5	$\mathbb{C}\otimes\mathbb{C}$	2	$2(\phi_1\pm\phi_i)$	1	$(A_1)^2$	$\phi_1\pm\phi_i$	2	$(BC_1)^2$
6	$\mathbb{C}\otimes\mathbb{H}$	2	$2(\phi_1\pm\phi_i)$	1				
			$2\phi_1, 2\phi_i$	2	B_2	$\phi_1\pm\phi_i$	4	BC_2
7	$\mathbb{C}\otimes\mathbb{O}$	2	$2(\phi_1\pm\phi_i)$	1				
			$2\phi_1, 2\phi_i$	6	B_2	$\phi_1\pm\phi_i$	8	BC_2
8	$\mathbb{H}\otimes\mathbb{H}$	4	$2(\phi_e\pm\phi_f)$	1	D_4	$\sum s_e\phi_e,\ \prod s_e=1$	4	B_4
9	$\mathbb{H}\otimes\mathbb{O}$	4	$2(\phi_e\pm\phi_f)$	1				
			$2\phi_e$	4	B_4	$\sum s_e\phi_e$	4	F_4
10	$\mathbb{O}\otimes\mathbb{O}$	8	$2(\phi_e\pm\phi_f)$	1	D_8	$\sum s_e\phi_e,\ \prod s_e=1$	1	E_8

TABLE 1. Roots of the Rosenfeld planes

13. Periodicity of Clifford representations via octonions

The irreducible representations S_n of Cl_n with $n \le 8$ are as follows. Let $B_n \subset B_{\mathbb{O}}$ denote the standard basis of $\mathbb{R}^n$.

n	0	1	2	3	4	5	6	7	8
S_n	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{O}=\mathbb{H}^2$	$\mathbb{O}=\mathbb{C}^4$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}^2$
$e\in B_n$	–	L_e	L_e	$\pm L_e$	$\begin{pmatrix} & -\bar e \\ e & \end{pmatrix}$	L_e	L_e	$\pm L_e$	$\begin{pmatrix} & -\bar e \\ e & \end{pmatrix}$
$\underline{Cl}_n$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{H}^{2\times 2}$	$\mathbb{C}^{4\times 4}$	$\mathbb{R}^{8\times 8}$	$\mathbb{R}^{8\times 8}$	$\mathbb{R}^{16\times 16}$

TABLE 2. Irreducible Clifford modules

The representation of Cl_2 is just a restriction of (any of) those for Cl_3, and the representations of Cl_4, Cl_5, Cl_6 are restrictions of (any of) those for Cl_7. The complex structure on S_5 is given by $L(e_6)L(e_7)$ which commutes with $L(e_1),\dots,L(e_5)$, and the quaternionic structure on S_4 comes from the anticommuting complex structures $L(e_5)L(e_7)$ and $L(e_6)L(e_7)$.

What happens for $n \ge 9$? Then we may repeat the construction of (8.1), replacing $\mathbb{L}$ (which is the irreducible Cl_{l-1}-module) by the irreducible Cl_n-module S_n for arbitrary n. Then we obtain an irreducible

Cl_{8+n}-module $\mathbb{O}^2 \otimes S_n$ by putting

$$\mathbb{O} \oplus \mathbb{R}^{n+1} \ni (x, \lambda) \mapsto \begin{pmatrix} \lambda & -\bar{x} \\ x & \bar{\lambda} \end{pmatrix} \tag{13.1}$$

where $\mathbb{R}^{n+1} = \mathbb{R}^n \oplus \mathbb{R} \cdot 1 \subset Cl_n$ and where the "conjugation" $\kappa : \lambda \mapsto \bar{\lambda}$ is the reflection at $\mathbb{R} \cdot 1$, that is $\kappa(1) = 1$ and $\kappa = -I$ on $\mathbb{R}^n$. As before, the Clifford relations (7.2) are immediate. Therefore we have the following periodicity of length 8 for the irreducible representations of Cl_n – recall that there is precisely one irreducible Cl_n-representation, up to equivalence and automorphisms:

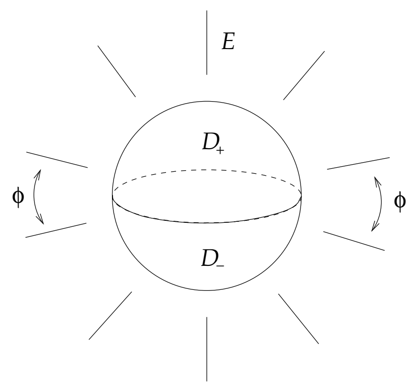
Theorem 13.1. *Periodicity Theorem for Clifford modules:*

$$S_{n+8} = S_n \otimes \mathbb{O}^2 \tag{13.2}$$

This theorem together with Table 2 determines all irreducible Clifford representations.

Remark. One could do the same with Cl_4 and $\mathbb{H}^2$ in place of Cl_8 and $\mathbb{O}^2$, but this would not give an irreducible representation. E.g. we obtain another Cl_8-module $\mathbb{H}^2 \otimes \mathbb{H}^2$ with dimension 64 while the irreducible module $\mathbb{O}^2$ has dimension 16. We have seen the reducibility in Lemma 9.1: the representation of $Spin_8$ decomposes into 4 irreducible subspaces. The representation of K becomes irreducible only through the $S_{\mathbb{H}} \cdot S'_{\mathbb{H}}$ factor. Thus the octonions are responsible for the 8-periodicity.

14. Vector bundles over spheres



(Sections 14-17 contain common work with Bernhard Hanke [6].) Clifford representations have a direct connection to vector bundles over spheres and hence to K-theory. Every vector bundle $E \to \mathbb{S}^{k+1}$ is trivial over each of the two closed hemispheres $D_+, D_- \subset \mathbb{S}^{k+1}$, but along the equator $\mathbb{S}^k = D_+ \cap D_-$ the fibres over ∂D_+ and ∂D_- are identified by some map $\phi : \mathbb{S}^k \to O_n$ called *clutching map.* Homotopic clutching maps define equivalent vector bundles. Thus vector bundles over $\mathbb{S}^{k+1}$ are classified by the homotopy group $\pi_k(O_n)$. When we allow for adding of trivial bundles (stabilization), n may be arbitrarily high. Let $\mathcal{V}_k$ be the set of vector bundles over $\mathbb{S}^{k+1}$ up to equivalence and adding of trivial bundles ("stable vector bundles"). Then

$$\mathcal{V}_k = \lim_{n\to\infty} \pi_k(O_n). \tag{14.1}$$

A Cl_k module $S = \mathbb{R}^n$ or the corresponding Clifford system $J_1, \dots, J_k \in O_n$ defines a peculiar map $\phi = \phi_S : \mathbb{S}^k \to O_n$ which is *linear*, that is a restriction of a linear map $\phi : \mathbb{R}^{k+1} \to \mathbb{R}^{n\times n}$, where we put

$$\phi_S(e_{k+1}) = I, \quad \phi_S(e_i) = J_i \text{ for } i \le k. \tag{14.2}$$

The bundles defined by such clutching maps ϕ_S are called *generalized Hopf bundles.* In the cases $k = 1, 3, 7$, these are the classical complex, quaternionic, and octonionic Hopf bundles over $\mathbb{S}^{k+1}$.

Remark. Cl_k-modules are in 1:1 correspondence to linear maps $\phi : \mathbb{S}^k \to O_n$ with the identity matrix in the image. In fact, let ϕ be such map and denote the linear extension $\mathbb{R}^{k+1} \to \mathbb{R}^{n\times n}$ by the same symbol. Then ϕ is isometric for the inner product $\langle A, B\rangle = \frac{1}{n}\,\mathrm{trace}\,(A^T B)$ on $\mathbb{R}^{n\times n}$ since it maps the unit sphere $\mathbb{S}^k$ into O_n which is contained in the unit sphere of $\mathbb{R}^{n\times n}$. For all $A, B \in \phi(\mathbb{S}^k)$ we have $(A + B) \in \mathbb{R}\cdot O_n$. On the other hand, $(A+B)^T(A+B) = 2I + A^TB + B^TA$, thus $A^TB + B^TA = tI$ for some $t \in \mathbb{R}$. From the inner product with I we obtain $t = 2\langle A, B\rangle$. Inserting $A = I$ and $B \perp I$ yields $B + B^T = 0$, and for any $A, B \perp I$ we obtain $AB + BA = -2\langle A, B\rangle I$. Thus $\phi|\mathbb{R}^k$ defines a Cl_k-representation on $\mathbb{R}^n$.

Atiyah, Bott and Shapiro [1] reduced the theory of vector bundles over spheres to the simple algebraic structure of Clifford modules by showing in particular:

Theorem 14.1. *All vector bundles over spheres are stably equivalent to generalized Hopf bundles.*

In fact, in [1] there is a refined version of this theorem which needs some notation. Let $\mathcal{M}_k$ be the set of equivalence classes of Cl_k-modules, modulo trivial Cl_k-representations. We have a map

$$\hat\alpha : \mathcal{M}_k \to \mathcal{V}_k$$

which assigns to each $S \in \mathcal{M}_k$ the corresponding generalized Hopf bundle over $\mathbb{S}^{k+1}$, viewed as a clutching map of a vector bundle. It is additive with respect to direct sums and onto by Theorem 14.1. But it is not 1:1. In fact, every Cl_{k+1}-module is also a Cl_k module simply because $Cl_k \subset Cl_{k+1}$. This defines a restriction map $\rho : \mathcal{M}_{k+1} \to \mathcal{M}_k$. Any Cl_k-module S which is in fact a Cl_{k+1}-module gives rise to a contractible clutching map $\phi_S : \mathbb{S}^k \to SO_n$ and hence to a trivial vector bundle since ϕ_S can be extended to $\mathbb{S}^{k+1}$ and thus contracted over one of the half spheres $D_+, D_- \subset \mathbb{S}^{k+1}$. Thus $\hat{\alpha}$ sends $\rho(\mathcal{M}_{k+1})$ into trivial bundles and hence it descends to an additive map[22]

$$\alpha : \mathcal{A}_k := \mathcal{M}_k/\rho(\mathcal{M}_{k+1}) \to \mathcal{V}_k.$$

In fact, $\mathcal{A}_k$ and $\mathcal{V}_k$ are abelian groups with respect to direct sum (not just semigroups). Moreover, $\mathcal{A} = \sum_k \mathcal{A}_k$ and $\mathcal{V} = \sum_j \mathcal{V}_k$ are graded rings with respect to the multiplication by tensor products. Here is the refined version of the Atiyah-Bott-Shapiro theorem [1]:

Theorem 14.2. *$\alpha : \mathcal{A} \to \mathcal{V}$ is a graded ring isomorphism.*

Recall from Table 2, page 34, the one or (for $k = 3, 7$) two generators S_k of $\mathcal{M}_k$.

k	0	1	2	3	4	5	6	7	8
S_k	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\pm\mathbb{H}$	$\mathbb{O}$	$\mathbb{O}$	$\mathbb{O}$	$\pm\mathbb{O}$	$\mathbb{O}^2$

From this we can easily deduce the groups $\mathcal{A}_k$. If $S_k = \rho(S_{k+1})$, then $\mathcal{A}_k = 0$. This happens for $k = 2, 4, 5, 6$. For $k = 0, 1$ we have

$$\rho(S_{k+1}) = S_k \oplus S_k = 2S_k,$$

hence $\mathcal{A}_0 = \mathcal{A}_1 = \mathbb{Z}_2$. For $k = 3, 7$ there are two generators for $\mathcal{M}_k$, say S_k and S'_k (denoted $\pm$ in the table), and $\rho(S_{k+1}) = S_k \oplus S'_k$, thus $\mathcal{A}_3 = \mathcal{A}_7 = \mathbb{Z}$. Hence

$$\begin{array}{c|cccccccc} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mathcal{A}_k & \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} \end{array} \tag{14.3}$$

and because of the periodicity we have $\mathcal{A}_{k+8} = \mathcal{A}_k$. Note also that by (14.1) we have for sufficiently large n:

$$\pi_k(O_n) = \mathcal{A}_k.$$

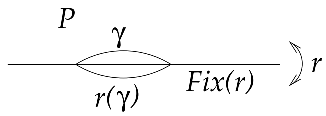
We will sketch a proof of this theorem which is different from that of [1]. We will use the original ideas of Bott and Milnor and deform the clutching map $\phi : \mathbb{S}^k \to G = SO_n$ of the given vector bundle $E \to \mathbb{S}^{k+1}$ step by step into a linear map. This will require some geometric ideas on symmetric spaces which are interesting for their own sake.

[22] In fact, both $\mathcal{V}_k$ and $\mathcal{A}_k$ are abelian groups with respect to direct sums, not just semigroups, and α is a group homomorphism. Using the tensor product, $\mathcal{V} = \sum_k \mathcal{V}_k$ and $\mathcal{A} = \sum_k \mathcal{A}_k$ become rings and α a ring homomorphism, see [1].

15. Poles and Centrioles

Recall that a *symmetric space* is a Riemannian manifold P with an isometric point reflection s_p (called *symmetry*) at any point $p \in P$, that is $s_p \in \hat{G}$ = isometry group of P with $s_p(\exp_p(v)) = \exp_p(-v)$ for all $v \in T_pP$. Prominent examples are compact Lie groups with bi-invariant metrics; the symmetry s_e at the unit element e is the inversion.

A *symmetric subspace* of a symmetric space P is a submanifold $Q \subset P$ with $s_q(Q) = Q$ for all $q \in Q$. They are totally geodesic, that is every Q-geodesic is also a P-geodesic.[23] Vice versa, any closed connected totally geodesic submanifold $Q \subset P$ is a symmetric subspace since s_q reverses any Q-geodesic through q and thus preserves Q. Examples of symmetric subspaces are connected components of the fixed point set of an isometry $r : P \to P$:



Otherwise short geodesic segments in the ambient space P with end points in a component of $\mathrm{Fix}(r)$ were not unique, see figure. If the isometry r is an involution ($r = r^{-1}$), its fixed point components are called *reflective*.

Any symmetric space is born with an equivariant map $s : p \mapsto s_p : P \to \hat{G}$ which is called *Cartan map*. Its image is a connected component of the set $\{g \in \hat{G} : g^{-1} = g\}$, the fixed set of the inversion. Hence $s(P) \subset \hat{G}$ is totally geodesic and $s : P \to s(P)$ is a covering of symmetric spaces.

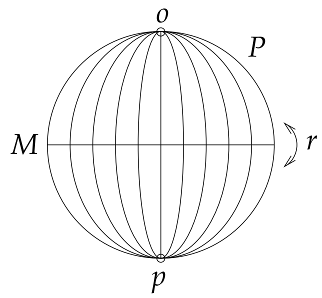
Two points $o, p \in P$ will be called *poles* if $s_p = s_o$. The notion was coined for the north and south pole of a round sphere, but there are many other spaces with poles; e.g. $P = SO_{2n}$ with $o = I$ and $p = -I$, or the Grassmannian $P = \mathsf{G}_n(\mathbb{R}^{2n})$ with $o = \mathbb{R}^n$ and $p = (\mathbb{R}^n)^\perp$. A geodesic γ connecting $o = \gamma(0)$ to $p = \gamma(1)$ is reflected into itself at o and p and hence it is closed with period 2.

Now we consider the *midpoint set* M between poles o and p,

$$M = \{m = \gamma\left(\tfrac{1}{2}\right) : \gamma \text{ geodesic in } P \text{ with } \gamma(0) = o,\ \gamma(1) = p\}.$$

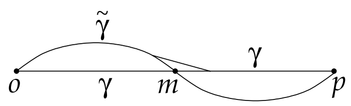
For the sphere $P = \mathbb{S}^n$ with north pole o, this set would be the equator, see figure below.

[23]Let γ be a Q-geodesic with curvature vector $\eta = \nabla_{\gamma'}\gamma'$ at some point $q \in Q$. Then s_q would preserve γ and η, but on the other hand, $(s_q)_*(\eta) = -\eta$ thus $\eta = 0$ and γ is also a P-geodesic.



The points o and p are identified by the Cartan map s. If γ is any geodesic with $\gamma(0) = o$ and $\gamma(1) = p$, it is reversed by s_o, and $s(\gamma)$ becomes a closed geodesic in $s(P)$ preserved by the symmetry in $s(P)$ at s_o (the conjugation with s_o). Applying s, the midpoint $m = \gamma(\frac{1}{2})$ becomes the point on $s \circ \gamma$ opposite to s_o, hence it is fixed under the symmetry at s_o. So the components of $s(M)$ are totally geodesic, and the same holds for M since "totally geodesic" is a local property. In fact, M itself is reflective: it is fixed by $\delta \circ s_p$ where δ is the deck transformation of the covering $s : P \to s(P)$ interchanging o and p.

Connected components of the midpoint set M are called *centrioles* [5]. These are most interesting when the corresponding geodesics between o and p are shortest ("*minimal centrioles*"). Each such midpoint $m = \gamma(\frac{1}{2})$ determines its geodesic γ uniquely: if there is another geodesic $\tilde\gamma$ from o to p through m, it can be made shorter by cutting the corner.



There exist chains of minimal centrioles (centrioles in centrioles),

$$P \supset P_1 \supset P_2 \supset \dots \tag{15.1}$$

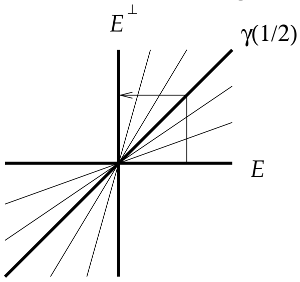
Peter Quast [16, 17] classified all such chains with at least 3 steps starting with a compact simple Lie group $P = G$. Up to group coverings, the result is as follows. The chains 1,2,3 occur in Milnor [11].

No.	G	P_1	P_2	P_3	P_4	restr.
1	$(S)O_{4n}$	SO_{4n}/U_{2n}	U_{2n}/Sp_n	$\mathsf{G}_p(\mathbb{H}^n)$	Sp_p	$p = \frac{n}{2}$
2	$(S)U_{2n}$	$\mathsf{G}_n(\mathbb{C}^{2n})$	U_n	$\mathsf{G}_p(\mathbb{C}^n)$	U_p	$p = \frac{n}{2}$
3	Sp_n	Sp_n/U_n	U_n/SO_n	$\mathsf{G}_p(\mathbb{R}^n)$	SO_p	$p = \frac{n}{2}$
4	$Spin_{n+2}$	Q_n	$(\mathbb{S}^1 \times \mathbb{S}^{n-1})/\pm$	$\mathbb{S}^{n-2}$	$\mathbb{S}^{n-3}$	$n \geq 3$
5	E_7	$E_7/(\mathbb{S}^1 E_6)$	$\mathbb{S}^1 E_6/F_4$	$\mathbb{OP}^2$	$-$	

TABLE 3. Chains of minimal centrioles

By $\mathsf{G}_p(\mathbb{K}^n)$ we denote the Grassmannian of p-dimensional subspaces in $\mathbb{K}^n$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Further, Q_n denotes the complex quadric in $\mathbb{CP}^{n+1}$ which is isomorphic to the real Grassmannian $\mathsf{G}_2^+(\mathbb{R}^{n+2})$ of oriented 2-planes, and $\mathbb{OP}^2$ is the octonionic projective plane $F_4/Spin_9$.

A chain is extendible beyond P_k if and only if P_k contains poles again. E.g. among the Grassmannians $P_3 = \mathsf{G}_p(\mathbb{K}^n)$ only those of half dimensional subspaces ($p = \frac{n}{2}$) enjoy this property: Then $(E, E^\perp)$ is a pair of poles for any $E \in \mathsf{G}_{n/2}(\mathbb{K}^n)$, and the corresponding midpoint set is the group $O_{n/2}, U_{n/2}, Sp_{n/2}$ since its elements are the graphs of orthogonal $\mathbb{K}$-linear maps $E \to E^\perp$, see figure below.



16. Minimal centrioles and Clifford modules

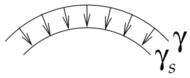
For compact matrix groups $P = G$ containing $-I$, there is a linear algebra interpretation for the iterated minimal centrioles P_j. Recall that a representation of Cl_k on a euclidean vector space V is given by k anticommuting orthogonal complex structures $J_1, \dots, J_k$ on V.

Theorem 16.1. *Let $G \subset GL(V)$ be a compact matrix group with $-I \in G$. Then a chain of minimal centrioles $G \supset P_1 \supset \cdots \supset P_k$ corresponds to a Cl_k-representation $J_1, \dots, J_k$ on V with $J_j \in G$, and each P_j is the connected component containing J_j of the set*

$$M_j = \{J \in G : J^2 = -I,\ JJ_i = -J_iJ \text{ for } i < j\}. \tag{16.1}$$

Proof. A geodesic γ in G with $\gamma(0) = I$ is a *one-parameter subgroup*, a Lie group homomorphism $\gamma : \mathbb{R} \to G$.[24] When $\gamma(1) = -I$, then $\gamma(\frac{1}{2}) = J$ is a *complex structure*, $J^2 = -I$. Thus the midpoint set M_1

[24]Let $\gamma : \mathbb{R} \to G$ be a smooth group homomorphism. Its curvature vector field $\eta = \nabla_{\gamma'}\gamma'$ is γ-invariant, $\eta(t) = \gamma(t)_*\eta(0)$. Thus the neighbour curve $\gamma_s(t) = \exp_{\gamma(t)}(s\eta(t))$ is another γ-orbit, $\gamma_s(t) = \gamma(t)g_s$. Deforming in the curvature vector direction shortens a curve, thus γ_s is shorter than γ on any finite intervall.



But on the other hand $\gamma_s = R(g_s)\gamma$ has the same length as γ since $R(g_s)$ is an isometry of G. Thus $\eta = 0$.

is the set of complex structures in G. When a connected component P_1 of M_1 contains antipodal points J_1 and $-J_1$, there is a next minimal midpoint set $M_2 \subset P_1$. It consists of points $\tilde{\gamma}(\frac{1}{2})$ where $\tilde{\gamma}$ is a shortest geodesic in P_1 from J_1 to $-J_1$. But P_1 ist totally geodesic in G, hence $\tilde{\gamma}$ is a geodesic in G too. Thus $\tilde{\gamma} = J_1\gamma$ where γ is a one-parameter subgroup in G with $\gamma(1) = -I$ which implies that $\gamma(\frac{1}{2}) =: J$ is a complex structure. Further, $\tilde{\gamma}(t) = J_1\gamma(t)$ is a complex structure for all t, that is $J_1\gamma J_1\gamma = -I$ or

$$J_1\gamma = \gamma^{-1}J_1. \tag{16.2}$$

In particular, J anticommutes with J_1. When γ is shortest in G, this condition $J_1J = -JJ_1$ is sufficient for (16.2): both $J_1\gamma$ and $\gamma^{-1}J_1$ are shortest geodesics from J_1 to $-J_1$ with midpoint $J_1J = -JJ_1$, so they must agree. Thus (16.2) is satisfied, that is $J_1\gamma$ takes values in P_1. In particular we see that geodesics from J_1 to $-J_1$ which are shortest in P_1 are also shortest in G. Now we put $J_2 = \tilde{\gamma}(\frac{1}{2}) = J_1J$; this is also a complex structure which anticommutes with J_1 and defines a connected component P_2 of M_2.

By induction hypothesis, we have anticommuting complex structures $J_i \in P_i$ for $i \le j$, and P_j is the connected component through J_j of the set M_j as in (16.1). Consider a geodesic $\tilde{\gamma}$ from J_j to $-J_j$ in P_j. Since P_j is totally geodesic, $\tilde{\gamma}$ is a geodesic in G, hence $\tilde{\gamma} = J_j\gamma$ where γ is a one-parameter group with $\gamma(1) = -I$. As before, $J_{j+1} := \tilde{\gamma}(\frac{1}{2}) = J_j\gamma(\frac{1}{2})$ is a complex structure which anticommutes with J_i for $i \le j$. When γ is shortest in G, these conditions imply $\tilde{\gamma}(t) \in M_{j+1}$ for all $t \in [0,1]$, since then $\tilde{\gamma}$ is determined by its midpoint. In particular, there are geodesics in P_j from J_j to $-J_j$ which are shortest in the ambient group G. Every connected component of the midpoint set of those geodesics defines a minimal centriole M_{j+1} satisfying (16.1) with j replaced by $j+1$. This finishes the induction step. □

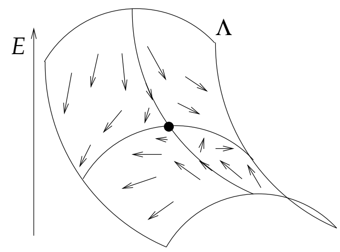
17. Deformation of clutching maps

Let E be a vector bundle over $\mathbb{S}^{k+1}$ and $\phi : \mathbb{S}^k \to G = SO_{2n}$ its clutching map. Since we are allowed to add trivial bundles, we may assume that n is large and divisible by a high power of 2. We declare $N = e_{k+1}$ to be the "north pole" of $\mathbb{S}^k$. First we deform ϕ such that $\phi(N) = I$ and $\phi(-N) = -I$. Thus ϕ maps each meridian from N to $-N$ in $\mathbb{S}^k$ onto some path from I to $-I$ in G, an element of the path space

$$\Lambda = \Lambda G = \{\lambda : [0,1] \overset{H^1}{\to} G : \lambda(0) = I,\ \lambda(1) = -I\}.$$

The meridians μ_v are labelled by $v \in \mathbb{S}^{k-1}$ where $\mathbb{S}^{k-1}$ is the equator of $\mathbb{S}^k$. Therefore ϕ can be viewed as a map $\phi : \mathbb{S}^{k-1} \to \Lambda G$. The matrix group G is equipped with a bi-invariant Riemannian metric, e.g. the trace metric. Using the negative gradient flow for the *energy function* E on the path space ΛG we may shorten all $\phi(\mu_v)$ simultaneously to minimal geodesics from I to $-I$ and obtain a map $\tilde{\phi} : \mathbb{S}^{k-1} \to \Lambda_o G$ where $\Lambda_o G$ is the set of shortest geodesics from I to $-I$, the minimum set of E on ΛG. This simultaneous shortening process is due to the *Morse theory* for the energy function E on Λ,

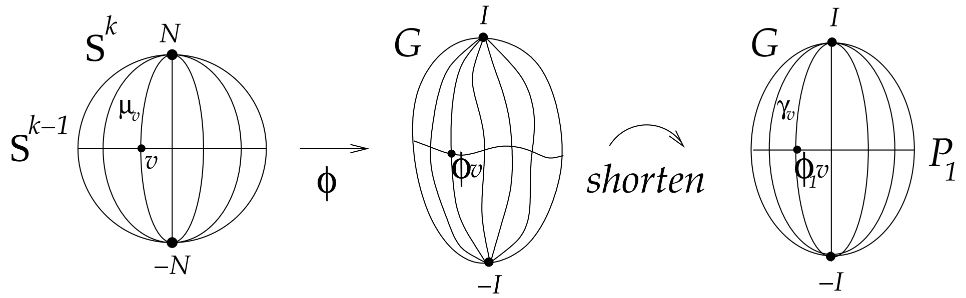
$$E(\lambda) = \int_0^1 |\lambda'(t)|^2 dt.$$



We may decrease the energy of any path λ by applying the gradient flow of $-E$. Most elements of Λ will be flowed to the minima of E which are the shortest geodesics between I and $-I$. The only exceptions are the domains of attraction ("unstable manifolds") for the other critical points, the non-minimal geodesics between I and $-I$. The codimension of the unstable manifold is the *index* of the critical point, the maximal dimension of any subspace where the second derivative of E (taken at the critical point) is negative. If β denotes the smallest index of all non-minimal critical points, any continuous map $f : X \to \Lambda$ from a connected cell complex X of dimension $< \beta$ can be moved away from these unstable manifolds and flowed into a connected component of the minimum set.

How large is β? In $G = SO_{2n}$, a shortest geodesic from I to $-I$ is a product of n half turns, planar rotations by the angle π in n perpendicular 2-planes in $\mathbb{R}^{2n}$. A non-minimal geodesic must make an additional full turn and thus a 3π-rotation in at least one of these planes, say in the x_1x_2-plane. This rotation belongs to the rotation group $SO_3 \subset SO_{2n}$ in the $x_1x_2x_k$-space for any $k \in \{3, \dots, 2n\}$. Using $SO_3 = \mathbb{S}^3/\pm$, we lift the 3π-rotation to $\mathbb{S}^3$ and obtain a 3/4 great circle which can be shortened. There are $2n-2$ coordinates x_k and therefore $2n-2$ independent contracting directions, hence the index of a nonminimal geodesic in SO_{2n} is $\geq 2n-2$ (compare [11, Lemma 24.2]).

A shortest geodesic $\gamma : [0,1] \to G$ from I to $-I$ is uniquely determined by its midpoint $m(\gamma)$. Using the midpoint map $m : \Lambda_o \to G$, the space of shortest geodesics from I to $-I$ can be viewed as the midpoint set $m(\Lambda_o) = M_1$, the set of complex structures in G. Thus we obtain a map $\phi_1 = m \circ \tilde{\phi} : \mathbb{S}^{k-1} \to P_1$, and we may replace ϕ by the geodesic suspension over ϕ_1 from I and $-I$.



Now we repeat this step replacing G by P_1 and ϕ by ϕ_1. Again we choose a "north pole" $N_1 = e_k \in \mathbb{S}^{k-1}$ and deform ϕ_1 such that $\phi_1(\pm N_1) = \pm J_1$ for some $J_1 \in P_1$. Now we deform the curves $\phi_1(\mu_1)$ for all meridians $\mu_1 \subset \mathbb{S}^{k-1}$ to shortest geodesics, whose midpoints define a map $\phi_2 : \mathbb{S}^{k-2} \to P_2$, and then we replace ϕ_1 by a geodesic suspension from $\pm J_1$ over ϕ_2. This step is repeated $(k-1)$-times[25] until we reach a map $\phi_{k-1} : \mathbb{S}^1 \to P_{k-1}$. This loop can be shortened to a geodesic loop $\tilde{\gamma} = J_{k-1}\gamma : [0,1] \to P_{k-1}$ (which is a closed geodesic since P_{k-1} is symmetric) starting and ending at J_{k-1}, such that $\tilde{\gamma}$ and γ are shortest in their homotopy class.

We have $\gamma(t) = e^{2\pi t A}$ for some skew-symmetric matrix A. From $\tilde{\gamma}^2 = -I$ and $J_i\tilde{\gamma} = -\tilde{\gamma}J_i$ for $i < k-1$ we obtain

$$AJ_{k-1} = -J_{k-1}A, \quad AJ_i = J_iA \ \text{ for } \ i < k-1. \tag{17.1}$$

In fact, $J_{k-1}e^{2\pi t A} = e^{-2\pi t A}J_{k-1} \Rightarrow J_{k-1}A = -AJ_{k-1}$ and $J_{k-1}e^{2\pi t A}J_i = -J_iJ_{k-1}e^{2\pi t A} \Rightarrow J_{k-1}AJ_i = -J_iJ_{k-1}A = J_{k-1}J_iA \Rightarrow AJ_i = J_iA$. Since γ is closed, the (imaginary) eigenvalues of A have the form $a\mathrm{i}$ with $a \in \mathbb{Z}$ and $\mathrm{i} = \sqrt{-1}$. Let $E_a \subset V \otimes \mathbb{C}$ be the corresponding eigenspace. Since every J_i, $i < k$ commutes or anticommutes with A, it keeps invariant $E_a + E_{-a}$ and also $\mathrm{Re}\,(E_a) = (E_a + E_{-a}) \cap V$. Now we split $\mathbb{R}^n$ into $V_0 = \ker A$ and a sum of subspaces V_j which are invariant under the linear maps $A, J_1, \dots, J_{k-1}$ and minimal with respect to this property. In particular $V_j \subset E_{a_j} + E_{-a_j}$ for some integer a_j. Hence on V_j we have

[25] One has to show yet that the index of nonminimal geodesics is large for all P_j, see [11, Section 24] or [6, Thm. 4]. The index was computed in a more general case by Mitchell [12, 13].

$A = a_j J$ for some complex structure J anticommuting with J_{k-1} and commuting with J_i, $i < k-1$, and

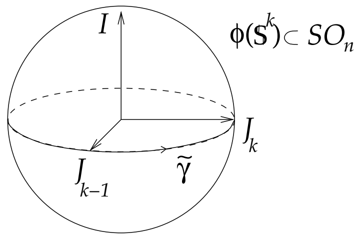
$$J_k := J_{k-1}J \tag{17.2}$$

is a complex structure which anticommutes with $J_1, \dots, J_{k-1}$. Hence V_j is an irreducible Cl_k-module with the dimension $m_k = \dim S_k$ independent of j. By the choice of the sign of a_j we can assume that all V_j are equivalent Cl_k-modules. In other words, $V = V_0 \oplus V'$ with $V' = V_1 \otimes \mathbb{R}^p$ where V_1 is an irreducible Cl_k module given by $J_1, \dots, J_k$ (living only on V_1) while $A = J \otimes \mathrm{diag}(a_1, \dots, a_p)$ with $J := J_k J_{k-1}$. If two of the integers a_i, a_j differ by two or more, we can use the p extra dimensions to shorten γ within P_{k-1}, see the subsequent Lemma 17.1. But this is impossible since γ is shortest in its homotopy class. Since we may always assume that, say, $a_1 = 0$ (by adding a trivial bundle if necessary), all nonzero a_j must be 1, (or all -1, but this is a matter of choice of J_k), thus $A = J$ on V'.

After this deformation our vector bundle E over $\mathbb{S}^{k+1}$ splits as $E = E_0 \oplus E'$ where E_0 is trivial and E' is the generalized Hopf bundle for the Clifford system $J_1, \dots, J_k$ on $V' = V_1 \otimes \mathbb{R}^p$, see subsequent remark.

Thus the map $\alpha : \mathcal{M}_k/\rho(\mathcal{M}_{k+1}) \to \mathcal{V}_k$ assigning to each Clifford module its generalized Hopf bundle is onto. It remains to show its injectivity which is done by Lemma 17.2 at the end of this section. This finishes the proof of Theorem 14.2.

Remark. We want to check explicitly that $\tilde\gamma(t) = J_{k-1}e^{2\pi t J}$ with $J = J_k J_{k-1}$ as defined in the previous proof is part of the linear map $\phi : \mathbb{S}^k \to SO_n$ defined by the Cl_k-representation $J_1, \dots, J_k$.

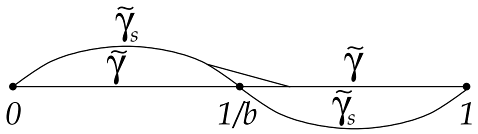


Recall that ϕ is mapping the unit basis $e_1, \dots, e_k, e_{k+1}$ of $\mathbb{R}^{k+1}$ onto $J_1, \dots, J_k, I \in SO_n$. The closed geodesic $\tilde\gamma : [0,1] \to SO_n$ is the restriction of ϕ to the intersection of $\mathbb{S}^k$ with the $e_{k-1}e_k$-plane, using the parametrization $x(t) = e_{k-1}\cos(2\pi t) + e_k \sin(2\pi t)$ for $t \in [0,1]$, more precisely, $\tilde\gamma(t) = \phi(x(t))$. It must satisfy (1) $\tilde\gamma(0) = J_{k-1}$ and (2) $\tilde\gamma(\frac{1}{2}) = -J_{k-1}$ while (3) $\tilde\gamma(\frac{1}{4}) = J_k$. Putting $\tilde\gamma(t) = J_{k-1}e^{2\pi t J}$ for some complex structure J, the first two conditions are satisfied while

$\tilde\gamma(\frac14) = J_{k-1}e^{(\pi/4)J} = J_{k-1}J$.[26] Hence condition (3) holds if and only if $J_k = J_{k-1}J$ as in (17.2) above.

Lemma 17.1. *If $a_i - a_j = b \geq 2$ for some $i, j \in \{1, \dots, p\}$, then the closed geodesic $\tilde\gamma(t) = J_{k-1}e^{2\pi t A}$ can shortened by a deformation in P_{k-1}.*

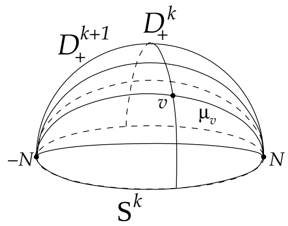
Proof. We have $V_i + V_j = V_1 \otimes \mathbb{R}^2$, and $A = J \otimes \operatorname{diag}(a_i, a_j)$ on $V_i + V_j$ where $J = J_k J_{k-1}$. Then $\operatorname{diag}(a_i, a_j) = \frac12(aI + bR_o)$ with $a = a_i + a_j$ and $R_o = \operatorname{diag}(1, -1)$. Now we deform A to A_s by replacing R_o with any planar reflection $R_s = \left(\begin{smallmatrix}\cos s & \sin s\\ \sin s & -\cos s\end{smallmatrix}\right)$ for $s \in [0, \pi]$, that is we replace $A = \frac12 J \otimes (aI + bR_o)$ by $A_s = \frac12 J \otimes (aI + bR_s)$ which still satisfies (17.1). Thus $\tilde\gamma_s(t) = J_{k-1}e^{2\pi t A_s} \in P_{k-1}$. Let $J_s = J \otimes R_s$ (with $J_s^2 = -I$). Then $\tilde\gamma_s(t) = J_{k-1}e^{\pi t a J}e^{\pi t b J_s}$. For $t = 1/b \leq 1/2$ we have $e^{\pi t b J_s} = e^{\pi J_s} = -I$, thus $\tilde\gamma_s(1/b)$ is independent of s. All the closed geodesics $\tilde\gamma_s$ must pass through the same point $\tilde\gamma_s(1/b)$, therefore they can be shortened by cutting the corner.



□

Lemma 17.2. *If a bundle $\hat\alpha(S)$ is trivial for some Cl_k-module S, then S is the restriction of a Cl_{k+1}-module.*

Proof. Let S be a Cl_k-module and $\phi = \phi_S : \mathbb{S}^k \to G$ the corresponding clutching map (that is $\phi(e_{k+1}) = I$, $\phi(e_i) = J_i$). We assume that ϕ is contractible, that is it extends to $\hat\phi : D^{k+1} \to G$. The closed disk D^{k+1} will be considered as the northern hemisphere $D_+^{k+1} \subset \mathbb{S}^{k+1}$. Repeating the argument above for the surjectivity, we consider the meridians μ_v between $N = e_{k+1} \in \mathbb{S}^k$ and $-N$, but this time there are much more such meridians, not only those in $\mathbb{S}^k$ but also those through the hemisphere D_+^{k+1}. They are labeled by $v \in D_+^k := D_+^{k+1} \cap N^\perp$.



[26]Note that for any complex structure J we have $e^{(\pi/4)J} = J$ since the eigenvalues are $\pm\mathrm{i}$ and $e^{\pm(\pi/4)\mathrm{i}} = \pm\mathrm{i}$.

Applying the negative energy gradient flow we deform the curves $\phi(\mu_v)$ to minimal geodesics without changing those in $\phi(\mathbb{S}^k)$ which are already minimal. Then we obtain the midpoint map $\hat\phi_1 : D^k_+ \to P_1$ with $\phi_1(v) = m(\hat\phi(\mu_v))$ which extends the given midpoint map ϕ_1 of ϕ. This step is repeated another $k-1$ times until we reach $\hat\phi_k : D^1_+ \to P_k$ which is a path from J_k to $-J_k$ in P_k. This path can be shortened to a minimal geodesic in P_k whose midpoint is a complex structure J_{k+1} anticommuting with $J_1, \dots, J_k$. Thus we have shown that our Cl_k-module S given by $J_1, \dots, J_k$ is extendible to a Cl_{k+1}-module, that is $S \in \rho(\mathcal{M}_{k+1})$. This finishes the proof of the injectivity.

18. Quaternionic exceptional symmetric spaces

Chains of minimal centrioles of a compact matrix group $G \subset GL(V)$ with $-I \in G$ are even more interesting [16] when V is a quaternionic vector space and G acts $\mathbb{H}$-linearly on V, in other words, it commutes with the action of $Sp_1 = \mathbb{S}^3 \subset \mathbb{H}$ by scalar multiplication on V. Using the quaternionic scalars i, j, k, anticommuting complex structures $J_1, J_2, J_3 \in G$ can be changed into commuting involutions $S_1 = iJ_1$, $S_2 = jJ_2$, $S_3 = kJ_3$. Then S_1 splits V into its half dimensional eigenspaces E_1 and $E_1^\perp = jE_1$ (such subspaces are called *totally complex*), preserved by S_2 and S_3. Further, S_2 splits E_1 into its half dimensional eigenspaces E_2 and $E_2^\perp = iE_2$ (such subspaces are called *totally real*), and S_3 splits E_2 into its eigenspaces E_3 and $E_3^\perp$ which may have arbitrary dimensions. Then $P_1 = \{E_1 \subset V\}^o$ (a connected component of all such subspaces E_1) and $P_2 = \{E_2 \subset E_1\}$ and $P_3 = \{E_3 \subset E_2\}$. An example is line 3 of Table 3, page 39, where $P_3 = \mathsf{G}_p(\mathbb{R}^n)$.

Let us now consider the case where V is also a Lie triple, more precisely, the Lie triple of a *quaternionic symmetric space*. A symmetric space P is called *quaternionic symmetric* if it is *quaternionic Kähler*, that is each tangent space is a quaternionic vector space, and the subspace $\mathbb{H} \subset \mathrm{End}(TP)$ of quaternionic scalar multiplications on any tangent space is parallel, and further the scalar multiplication with any unit scalar $q \in \mathbb{S}^3 \subset \mathbb{H}$ extends to a global isometry of P.

Remark. *Recall that for any Riemannian manifold P, the space $\mathbb{R}$ of real scalar multiplications is parallel, and P is symmetric if the real unit scalars $\pm 1 \in \mathbb{R}$ extend to isometries of P. It is hermitian symmetric or Kähler symmetric if it is Kähler, that is each tangent space carries a complex structure and the complex scalar multiplication is parallel, and further the complex unit scalars $\zeta \in \mathbb{S}^1 \subset \mathbb{C}$ extend to isometries of P. Quaternionic symmetric spaces are the quaternionic analogues.*

If P is quaternionic symmetric and K the isotropy group of some base point $o \in P$, then conjugation with any $k \in K$ preserves $Sp_1 \subset K$, thus $Sp_1 \subset K$ is a normal subgroup. On the Lie algebra level, $\mathfrak{sp}_1 \subset \mathfrak{k}$ and its orthogonal complement are ideals, hence we have a decomposition into ideals $\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{sp}_1$ and $K = K'Sp_1$ where K' commutes with Sp_1, that is, it acts $\mathbb{H}$-linearly on $V = T_oP$. We put $G = K'$ in the above construction. Then the spaces E_1, E_2, E_3 are fixed spaces of Lie triple automorphisms and hence subtriples, and we may characterize P_j as a connected component of certain Lie subtriples (totally complex, totally real, or a splitting of the latter).

Among the exceptional symmetric spaces there are 4 quaternionic symmetric spaces, namely $\{\mathbb{KHP}^2 \subset \mathbb{KOP}^2\}$ (the set of all symmetric subspaces congruent to $\mathbb{KHP}^2 \subset \mathbb{KOP}^2$) for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. As coset spaces these are the types (see [9, p. 518], [2, p. 313f]) $FI = F_4/Sp_3Sp_1$, $EII = E_6/SU_6Sp_1$, $EVI = E_7/Spin_{12}Sp_1$, $EIX = E_8/E_7Sp_1$ with quaternionic dimension $d = 7, 10, 16, 28$. The last example appears in Table 3, line 5 on page 39, while the others are in lines 1 - 3 for $n = 3$. The corresponding chains end with $\mathbb{KP}^2 = \mathsf{G}_1(\mathbb{K}^3)$, see also [16, p. 47].

We finish our lecture with some open questions related to this construction which combines all subjects treated in our lecture.

(1) What are the symmetric subspaces with Lie triples E_1, E_2, E_3?
(2) Are there other connected components?
(3) How can we geometrically understand the description of $\mathbb{KP}^2$ as $\{E_3 \subset E_2\}$?
(4) What is the relation to Freudenthal's magic squares? The numbers $d - 1$ appear in Freudenthal's work, see e.g. [8, p. 448].

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