Comparison Theorems in Riemannian Geometry

Figures

Fig. 1.

Fig. 2.

Fig. 3.
second time and let $S_2 = S \setminus S_1$. All the points above $S_1$ are in $V$ since they are closer to $S_1$ than to $\{x_n = -r\}$. Moreover, all points above $S_2$ are in $V$ for the same reason.

Fig. 25.

Thus we may connect any point of $S$ to some point in $\mathbb{R}^n_+ \cdot e_n$ within the shaded region (cf. fig. 26 below); we just have to avoid the cylinder of height $r$ above $\partial S$ if we start from $S_2$. This finishes the proof of the claim, of the lemma and of the theorem.

Fig. 26.

Moreover, due to $\text{Ric} \geq n - 1$ and the average comparison theorem 4.1, we have on $M \setminus \{p, q\}$:

$$\Delta \rho_p \leq (n - 1) \cot \rho_p, \quad \Delta \rho_q \leq (n - 1) \cot \rho_q$$

in the sense of support functions. In fact, to prove the first inequality at some point $x \in M \setminus \{p, q\}$, we choose a shortest geodesic segment $\beta$ from $x$ to $p$ and replace $p$ by some point $p'$ on $\beta$ close to $p$; then the distance function $\rho_{p'}$ from $p'$ is smooth near $r$ and satisfies the above inequality with an arbitrary small error (by Theorem 4.1).