THE PENROSE DECAGON

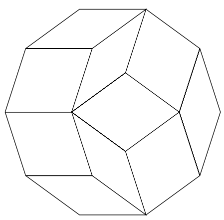
J.-H. ESCHENBURG AND H.J. RIVERTZ

Abstract. We display a Penrose tiling of the entire euclidean plane which is not of projection type although every compact subset is. It is related to 400 year old tilings in Iranian art.

1. Introduction

In 1972, Roger Penrose [6, 7] introduced a class of aperiodic tilings of euclidean plane which was based on the geometry of the regular pentagon. Some years later, de Bruijn [1] gave a different description of Penrose tilings as orthogonal projections of subsets ("strips") of the regular 5-grid $\mathbb{Z}^5 \subset \mathbb{R}^5$ onto a certain 2-plane $E \subset \mathbb{R}^5$. In the present paper we want to show that de Bruijn's construction does not exhaust all Penrose tilings; in fact, we will display a Penrose tiling which is not of projection type.

We need to show how to construct Penrose tilings of the entire plane without the projection method (section 2). This description is new. We show by a counterexample that it does not suffice to provide tiles and matching rules stating which pairs of oriented edges of the tiles may be adjacent to each other, such as in [10]. In section 3 we introduce a new Penrose tiling of the full plane. This arises from repeatedly subdividing the non-symmetric decagon ("Penrose decagon") occurring in every Penrose tiling all over the place.



We will call it "extended Penrose decagon" D. In section 4 we will show that this tilling cannot be obtained by the projection method. Before, we need to recall the geometry of projection tilings, sections 4 and 5. The idea to consider the extended Penrose ball came from the study of a certain

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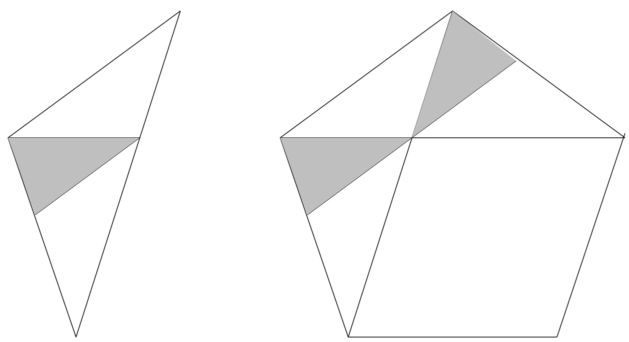
traditional pattern at Isfahan, Iran. We explain this connection in section 7. We conclude our paper by a short discussion of our main result.

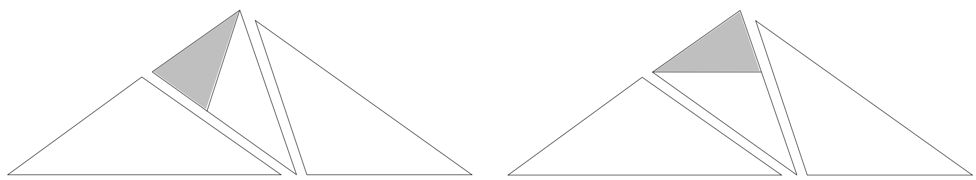
2. Penrose tilings of the full plane

To construct all Penrose tilings which extend to the full plane, start with a regular pentagon with two diagonals enclosing one sides. Draw a line segment parallel to the enclosed side through the intersection point of the diagonals. This bounds two narrow triangles which are coloured as in the subsequent figure. Now the two diagonals cut off two equisceles triangles, a broad one and a narrow one, which come with a subdivision by similar triangles, scaled down by the inverse golden ratio $1/\tau$.

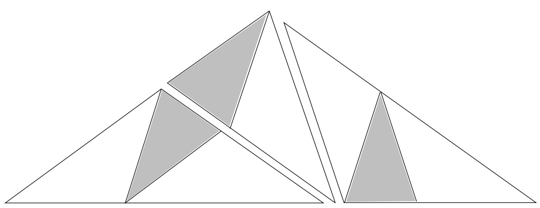


Since the narrow triangle is a subset of the broad one, we only consider the broad triangle with its subdivision. This is our initial (finite) Penrose tiling T_o where the tiles are the small triangles.

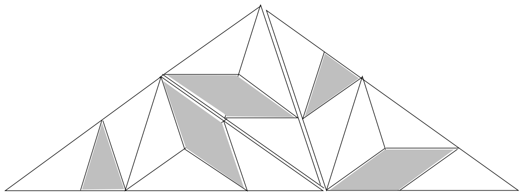
We may subdivide the small triangles in the same way as before. There are always two mirror symmetric ways to do this. But insisting that along each edge the subdivisions from both sides agree, we get uniqueness. In fact, the narrow tile in the middle may have one of the following two subdivisions:



However, only the left subdivision can be extended over the right edge of the narrow tile since both acute angles of the broad tile are uncoloured. Hence the subdivisions of the neighboring broad tiles are already fixed by the subdivision of the edge adjacent to the narrow tile:

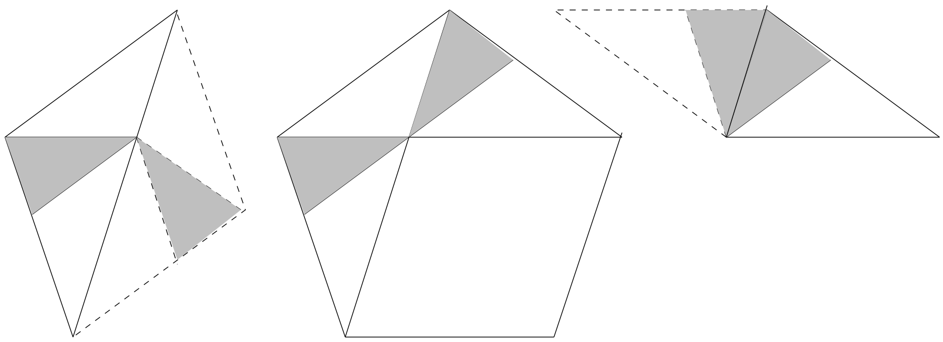


Call T_{n+1} the tiling which arises from T_n by subdiding all tiles and enlarging by the factor τ; then the tiles in T_{n+1} have the same size as the tiles in T_n. The tiling T_1 is in the figure above, and T_2 looks as follows.

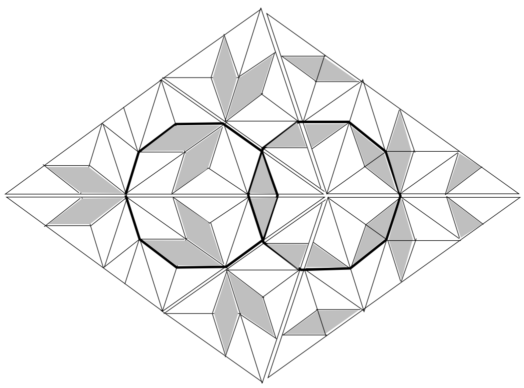


Definition. An *extendable Penrose tiling* is a tiling composed of these two types of tiles, the broad and the narrow triangle in the pentagon, such that any finite subset of tiles is isometric to a subset of some T_n.

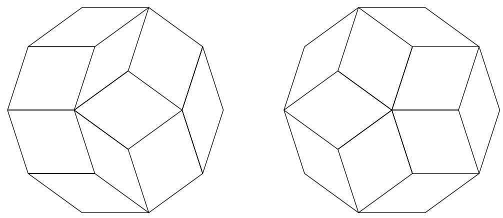
Since the third edge of any of the two equisceles triangles can be adjacent only to the third edge of a mirror symmetric copy of itself, the triangles always compose to rhombs in the interior of every Penrose tiling.



Using this doubling, our third Penrose tiling T_3 looks as follows.

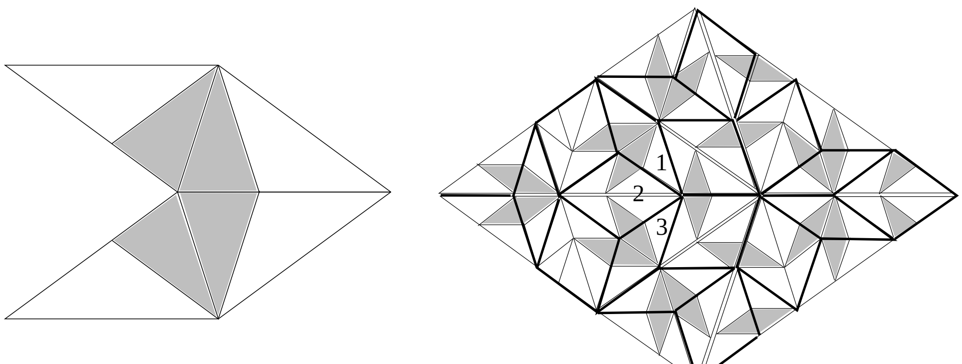


We have marked two decagons in the tiling where the right one is decagon-symmetric while the left one is only reflection symmetric.



Repeatedly subdividing the right decagon which we call *symmetric decagon*, we construct the two Penrose tilings with full decagonal symmetry. The other decagon, on the left, will be called *Penrose decagon*. It turns out that the interior part of its first subdivision is similar to the original tiling, see figure 1 below. Thus considering the union of all iterated subdivisions, each one scaled by τ and reflected at the vertical axis, we obtain a tiling of the full plane as explained in the next section. We will show that this tiling cannot be obtained by the projection method.

We still need to explain why matching rules never give sufficient conditions for unlimited extendability. The following figure shows the simplest situation which is build legally in terms of matching rules but cannot be extended.



The right figure which is the subdivision of T_2 (fat lines) shows why the left figure is legal with respect to any matching rules: the broad tile 2 is symmetric and bounded by the narrow tiles 1 and 3, hence 1 and 3 could be also adjacent to each other as shown in the left figure. But the left figure cannot be extended over the left boundary since the coloured vertices of the two tiles do not fit into the gap. Of course, the same problem may arise after an arbitrary number of subdivisions of the two tiles.

3. The extended Penrose decagon D

Subdivision of the Penrose decagon gives the pattern shown in figure 1 below. It is remarkable that a copy of the original Penrose decagon also occurs in the center of the subdivided pattern. Therefore, by successively subdividing and rescaling we get a tiling of the entire plane that is invariant

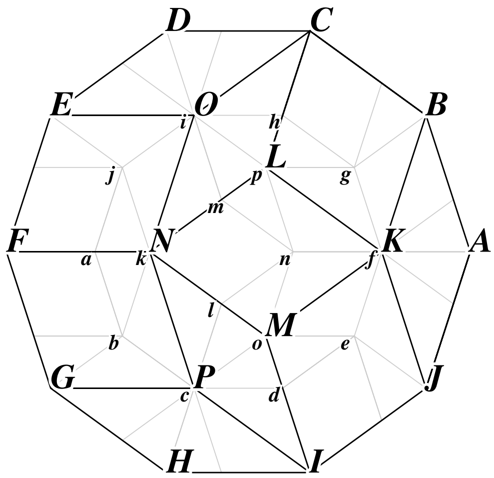


FIGURE 1. Subdivision of the Penrose decagon.

under subdivision. In fact, we start with the Penrose decagon D_1, subdivide, reflect at the vertical axis through the center and rescale by the golden ratio τ; then the points k, l, m, n in figure 1 are mapped onto K, L, M, N. Thus we obtain a new tiling D_2 of a τ-times larger decagon which contains D_1. It is clear that we can repeat this process arbitrarily: Given D_k, we obtain D_{k+1} by subdividing, reflecting and enlarging by τ, and $D_{k+1} \supset D_k$. The tiling which we are considering is the union of all D_k,

$$D = \bigcup_{k \in \mathbb{N}} D_k,$$

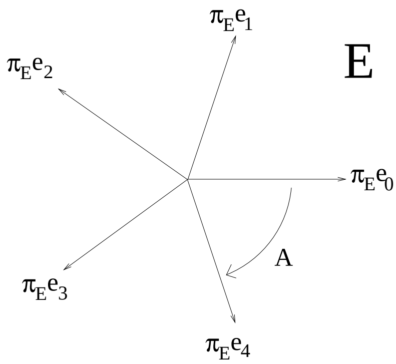
which certainly satisfies our definition since any finite subset lies in some D_k which in turn is a subset of some T_n; recall $D_1 \subset T_3$.

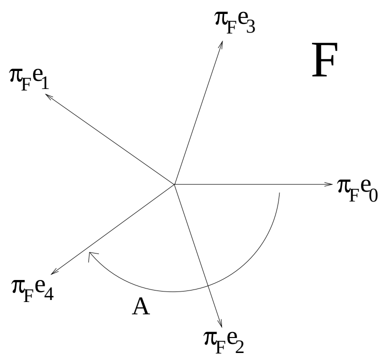
Theorem 1. *The Penrose tiling D is not of projection type.*

The rest of this article is devoted to the proof of this theorem.

4. Projection tilings

The tilings produced by the projection method [1] arise as follows. We consider the cyclic permutation $A = (12345)$ as an orthogonal matrix permuting the 5 coordinates of $\mathbb{R}^5$. It decomposes $\mathbb{R}^5$ orthogonally as $\mathbb{R}^5 = \mathbb{R}\mathbf{d} + E + F$ where $\mathbf{d} = (1,1,1,1,1)^T$ is a fixed vector and E, F are two invariant planes on which A acts by rotations of 72 and 144 degrees, respectively.





Let $\mathbf{a} \in \mathbb{R}^5$ such that $a := \langle \mathbf{a}, \mathbf{d} \rangle$ is an integer and such that no point of $E+\mathbf{a}$ has more than 2 integer coordinates; this property of $\mathbf{a}$ is called *general position.* Then the E-projection of the set

$$\Sigma_{\mathbf{a}} = \big((0,1)^5 + E + \mathbf{a}\big) \cap \mathbb{Z}^5$$

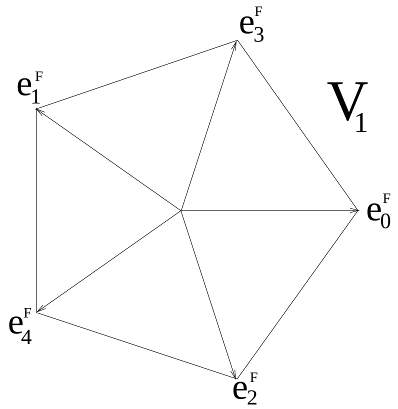
is the vertex set of a tiling $T_{\mathbf{a}}$ on E, and this is a Penrose tiling in the sense of our definition [1, 2, 9]. The elements of $\Sigma_{\mathbf{a}} \subset \mathbb{Z}^5$ are called *admissible* for the tiling $T_{\mathbf{a}}$. This condition can be transformed into planar geometry. First note first that any grid point $\mathbf{x} \in \mathbb{Z}^5$ lies on one of the hypersurfaces

$$H_k = \{\mathbf{x} \in \mathbb{R}^5 : \langle \mathbf{x}, \mathbf{d} \rangle = k\}$$

for some $k \in \mathbb{Z}$. The admissablility of a point $\mathbf{x} \in \mathbb{Z}^5 \cap H_k$ is decided by the so called *window* [1]

$$V_k = \pi_F\left((0,1)^5 \cap H_k\right) \tag{1}$$

which is nonempty only for $k \in \{1,2,3,4\}$; the subsequent figure shows V_1.



More precisely, for any point $\mathbf{x} \in \mathbb{Z}^n \cap H_k$ and for $a = \langle \mathbf{a}, \mathbf{d} \rangle$ we have

$$\mathbf{x} \in \Sigma_{\mathbf{a}} \iff \pi_F(\mathbf{x}) \in \pi_F(\mathbf{a}) + V_{k-a}\,. \tag{2}$$

In fact, note that $\mathbf{x} \in \Sigma_{\mathbf{a}} \iff \mathbf{x} \in \big((0,1)^5 + E + \mathbf{a}\big) \cap H_k \iff$

$$\pi_F(\mathbf{x}) \in \pi_F\left(\big((0,1)^5 + \mathbf{a}\big) \cap H_k\right) = \pi_F(\mathbf{a}) + V_{k-a},$$

since $\mathbf{x} = \mathbf{x}^o + \mathbf{a} \in H_k \iff \mathbf{x}^o \in H_k - \mathbf{a} = H_{k-a}$. For any $\mathbf{x} \in \Sigma_{\mathbf{a}}$ we denote

$$\operatorname{ind} \mathbf{x} := \langle \mathbf{x} - \mathbf{a}, \mathbf{d} \rangle = \langle \mathbf{x}, \mathbf{d} \rangle - a$$

the *index* of $\mathbf{x}$ and $\pi_E(\mathbf{x})$. In particular, when $\mathbf{a} = \frac{a}{5}\mathbf{d}$ for $a \in \{1, 2, 3, 4\}$, the tiling has the full pentagonal symmetry. Then $\mathbf{d} \in \Sigma_{\frac{a}{5}\mathbf{d}}$, and the symmetry center $\pi_E(\mathbf{d}) \in T_{\frac{a}{5}\mathbf{d}}$ has index $5 - a$. Note that $-I$ maps the cases $a = 1, 3$ isometric onto $a = 4, 2$, respectively. Figure 2 below shows the case $\mathbf{a} = \frac{1}{5}\mathbf{d}$ where the symmetry center has index 4.

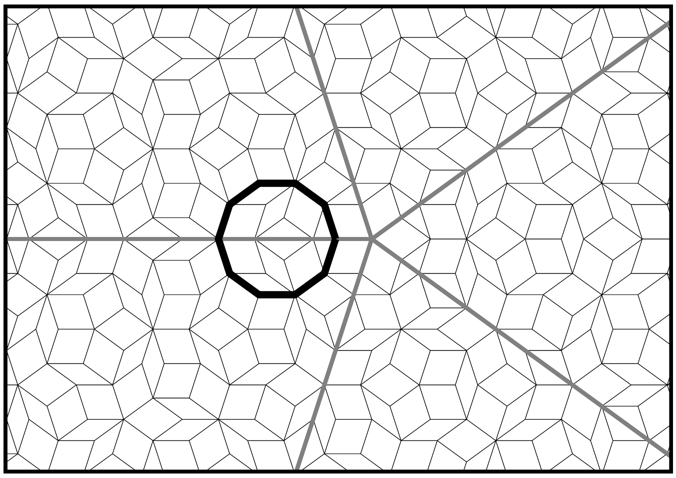
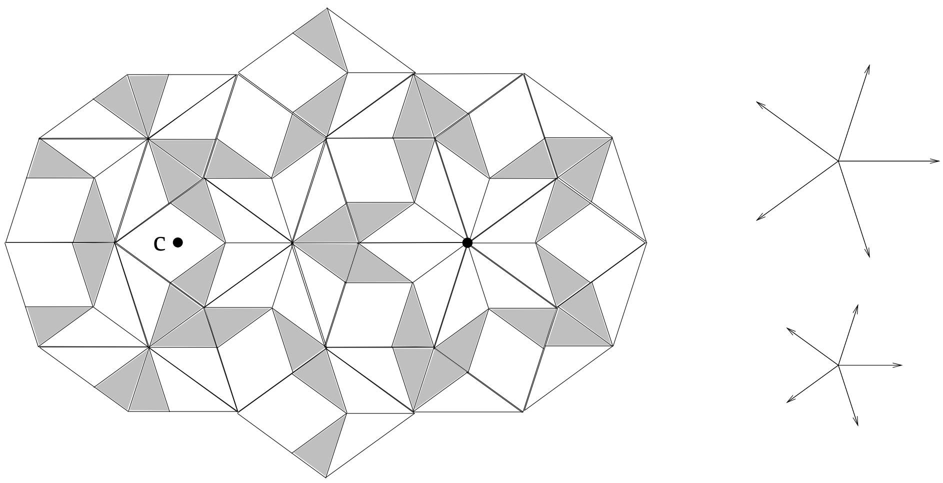


FIGURE 2. A Penrose decagon located in $T_{\frac{1}{5}\mathbf{d}}$.

We have marked a Penrose decagon in this tiling. Its center is located at $\pi_E(\mathbf{c})$ with $\mathbf{c} = \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_0$; its mirror image $\pi_E(-\mathbf{e}_2 - \mathbf{e}_3 + \mathbf{e}_0)$ is a vertex of the tiling. The next figure shows the subdivision which forms the other tiling with full pentagonal symmetry.



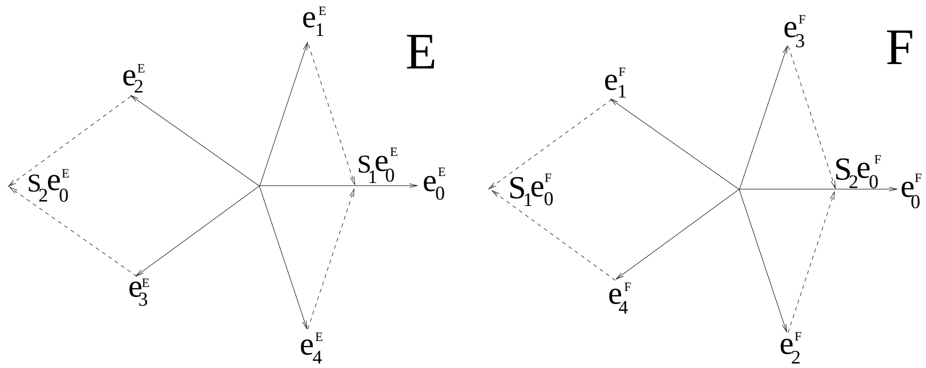
On the right margin we see the vectors $\pi_E(\mathbf{e}_i)$. Advancing by each of these vectors increases the index by one. Since the index can move only between 1 and 4, the symmetry center in the right decagon must have index 4 for the coarse tiling (long arrows) and index 2 for the fine tiling (short arrows). We see again the small Penrose decagon in the fine tiling inside the large one in the coarse tiling. However, after blowing up the subdivision by the factor τ, the center of the Penrose decagon is moved to the right. Therefore we change the origin to $\pi_E(\mathbf{c})$ by translating $T_{\mathbf{d}/5}$ to $T_{-\mathbf{c}+\mathbf{d}/5} = T_{\mathbf{d}/5} - \pi_E(\mathbf{c})$.

5. Subdivision of projection tilings

The subdivision of a projection tiling can be obtained from a linear map S on $\mathbb{R}^5$ with eigenspaces E, F and $\mathbb{R}\mathbf{d}$ which contracts on E and expands on F, thus broadening the strip and making more integer grid points admissible, see [3] for details. We use the A-invariant linear maps

$$S_k : \mathbf{e}_j \mapsto \mathbf{e}_{j+k} + \mathbf{e}_{j-k}$$

for $k = 1, 2$ where the indices are computed modulo 5. We have $S_k\mathbf{d} = 2\mathbf{d}$, and E, F are eigenspaces for S_1, S_2 with eigenvalues $1/\tau, -\tau$ on E and $-\tau, 1/\tau$ on F, respectively, as can be read from the following figure:



The maps $S = -S_1$ and $R = S_2$ are inverse to each other modulo $\mathbf{d}$ since $SR(\mathbf{e}_0) = S(\mathbf{e}_3 + \mathbf{e}_2) = -(\mathbf{e}_2 + \mathbf{e}_4 + \mathbf{e}_1 + \mathbf{e}_3) = -\mathbf{d} + \mathbf{e}_0$, and similar $SR(\mathbf{e}_j) = -\mathbf{d} + \mathbf{e}_j$. Moreover, each integer grid point $\mathbf{x} \in \mathbb{Z}^5$ lies on one of the hypersurfaces $H_k = \{\mathbf{x} \in \mathbb{R}^5 : \langle \mathbf{x}, \mathbf{d}\rangle = k\}$ for some $k \in \mathbb{Z}$, and S maps $H_k \cap \mathbb{Z}^5$ bijectively onto $H_{-2k} \cap \mathbb{Z}^5$. Since S expands on F by the factor τ, we have $S(V_k) \supset V_{k'}$ for $k' \equiv -2k$ mod 5 (see (1)) and hence $S(\Sigma_\mathbf{a} \cap H_k) \supset \Sigma_{S\mathbf{a}} \cap H_{k'}$ and

$$\pi_E S(\Sigma_\mathbf{a} \cap H_k) \supset \pi_E(\Sigma_{S\mathbf{a}} \cap H_{k'})\,. \tag{3}$$

This shows that the vertex set of the tiling $S(T_\mathbf{a})$ (which is $T_\mathbf{a}$, scaled down by the factor $-1/\tau$) contains the vertex set of the tiling $T_{S\mathbf{a}}$; in fact $S(T_\mathbf{a})$ is the first subdivision of $T_{S\mathbf{a}}$.

In particular, $S(\frac{1}{5}\mathbf{d}) = -\frac{2}{5}\mathbf{d} \equiv \frac{3}{5}\mathbf{d}$ mod $\mathbf{d}$, and $S(\frac{3}{5}\mathbf{d}) = -\frac{6}{5}\mathbf{d} \equiv \frac{4}{5}\mathbf{d}$ mod $\mathbf{d}$, hence S maps each of the two symmetric tilings $T_{\mathbf{d}/5}$ and $T_{3\mathbf{d}/5}$ onto the first subdivision of the other, up to sign. The same is true for the translated tilings $P = T_{-\mathbf{c}+\mathbf{d}/5}$ and $Q = T_{-\mathbf{c}+3\mathbf{d}/3}$: if we let $P_0 = P$ and $Q_0 = Q$ and define recursively P_{k+1}, Q_{k+1} as the first subdivision of P_k, Q_k, respectively, then

$$P_{k+1} = S(Q_k), \quad Q_{k+1} = -S(P_k). \tag{4}$$

We will also consider the inverse map $R = S_2$ which expands on E by the factor $-\tau$ and maps P_{k+1} onto Q_k and Q_{k+1} onto $-P_k$. In particular, the Penrose decagon $D_k \subset P_k$ (centered at the origin) is mapped onto the enlarged Penrose decagon $D_{k+1} \supset D_k$.

6. The tiling D does not fit into the window

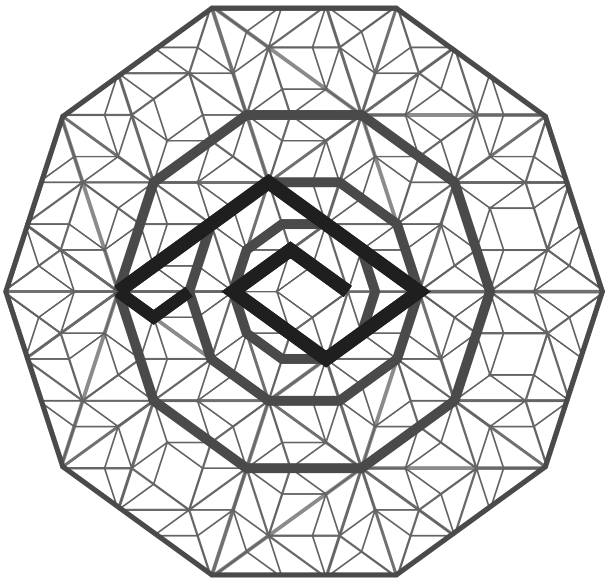
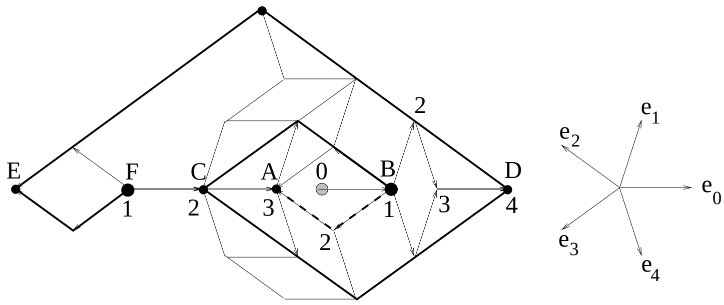


Figure 3. A path in D_3

In the figure, a path inside the third subdivision D_3 of the Penrose ball $D_0 \subset P_0$ is marked. Both of its end point B, F have index 1, as the following figure shows.



We obtain the broken line DE by applying three times the mapping R to the small broken line AB, and we reach the final point F by adding $-(\mathbf{e}_2 + \mathbf{e}_3)$. Hence we may describe the transition from the initial point $B = \pi_E(\mathbf{e}_0)$ to the final point F by the affine map $T(\mathbf{x}) = R^3\mathbf{x} + \mathbf{b}$ with $\mathbf{b} = -(\mathbf{e}_2 + \mathbf{e}_3) - \mathbf{d}$. (The meaning of the additional term $-\mathbf{d}$ which projects to 0 on E will become clear below.) Then we have $F = T(B)$. We iterate this process by putting $x_0 = \mathbf{e}_0$ and $\mathbf{x}_{j+1} = T(\mathbf{x}_j)$, then $\pi_E(\mathbf{x}_0) = B$ and $\pi_E(\mathbf{x}_1) = F$. In every step the path is prolongated, see figure 4 for $\mathbf{x}_2 = T^2\mathbf{x}_0 = R^6\mathbf{e}_0 + R^3\mathbf{b} + \mathbf{b}$. Note that all points $\pi_E\mathbf{x}_j$ lie on the horizontal line and have index 1.

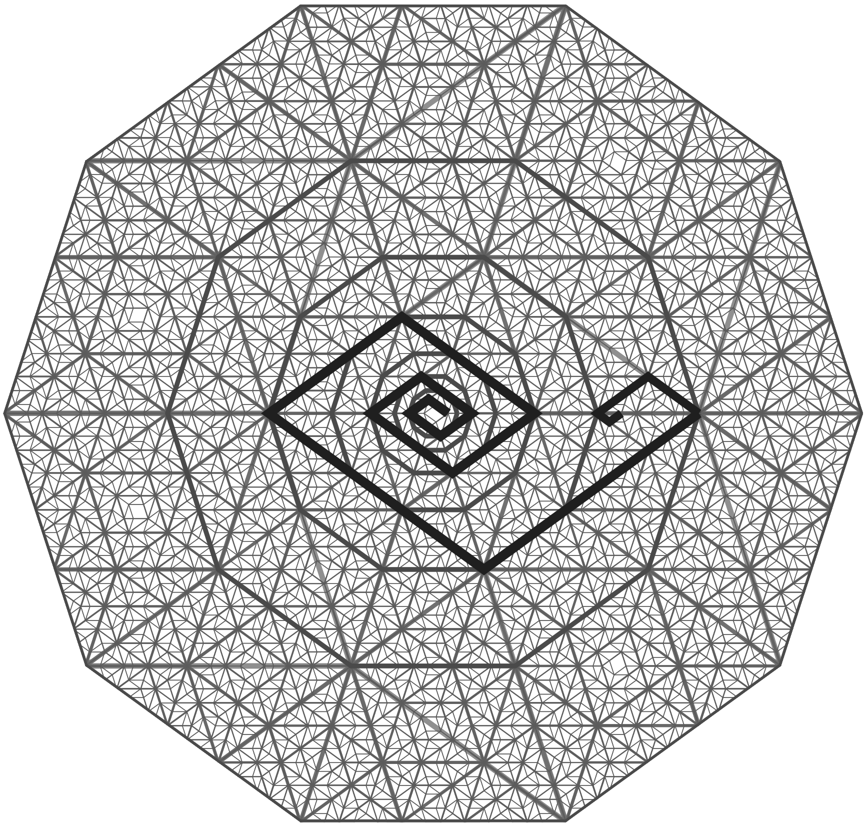


FIGURE 4

Now we want to show that the F-projection of the path $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ does not fit into the window V_1 since its length $\lim\limits_{j\to\infty} |\pi_F(\mathbf{x}_j) - \pi_F(\mathbf{x}_0)|$ is precisely the diameter of V_1 along the horizontal axis, $L = |\frac{1}{2}(\mathbf{e}_1^F + \mathbf{e}_4^F) - \mathbf{e}_0^F|$ (which is $(1 + \frac{1}{2}\tau)|\mathbf{e}_0^F|$), see the figure after (1). If D were a projection tiling, $\pi_F(\mathbf{x}_0)$

would have to lie somewhere in the open window V_1, thus for some $\epsilon > 0$, every point $\pi_F(\mathbf{x}_j)$ must have distance $\leq L - \epsilon$ from $\pi_F(\mathbf{e}_0)$, a contradiction.

To compute the length of the path on F we show first that

$$\mathbf{f} := \tfrac{1}{2}(\mathbf{e}_1 + \mathbf{e}_4) = \tfrac{1}{2} S_1 \mathbf{e}_0$$

is a fixed vector for the affine map $T = S_2^3 + \mathbf{b}$ on $\mathbb{R}^5$. In fact,

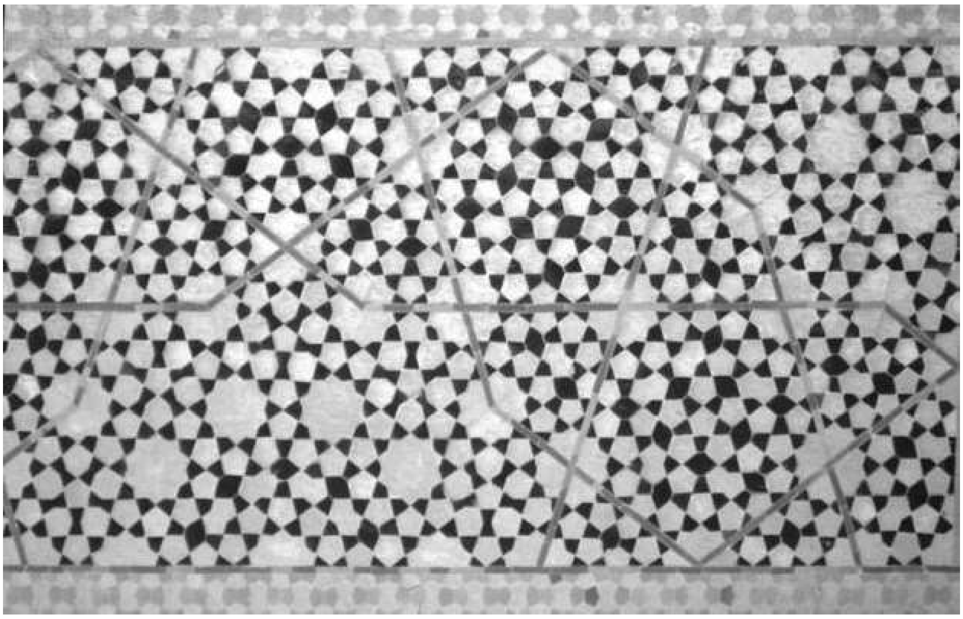
$$\begin{aligned}
S_2^3 \mathbf{f} + \mathbf{b} &= \tfrac{1}{2} S_2^3 S_1 \mathbf{e}_0 + \mathbf{b} \\
&= \tfrac{1}{2} S_2^2 (\mathbf{d} - \mathbf{e}_0) - S_2 \mathbf{e}_0 - \mathbf{d} \\
&= \tfrac{1}{2}(2I + S_1)(\mathbf{d} - \mathbf{e}_0) - (S_2 + S_1)\mathbf{e}_0 + S_1 \mathbf{e}_0 - \mathbf{d} \\
&= \mathbf{d} - \mathbf{e}_0 + \tfrac{1}{2} S_1 (\mathbf{d} - \mathbf{e}_0) - \mathbf{d} + \mathbf{e}_0 + S_1 \mathbf{e}_0 - \mathbf{d} \\
&= \tfrac{1}{2} S_1 \mathbf{e}_0 \ = \ \mathbf{f}
\end{aligned}$$

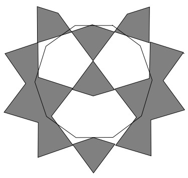
using $S_2 S_1 \mathbf{e}_0 = (S_2 + S_1)\mathbf{e}_0 = \mathbf{d} - \mathbf{e}_0$ and $S_2^2 = 2I + S_1$ and $S_1 \mathbf{d} = 2\mathbf{d}$.

Hence the affine map $T_F := \pi_F \circ T|_F$ has the fixed point $\pi_F(\mathbf{f})$. The linear part S_2^3 of T has eigenvalue $1/\tau^3 < 1$ on F, hence T_F is a strong contraction. Thus the sequence $\pi_F(\mathbf{x}_j) = \pi_F(T^j(\mathbf{x}_0)) = T_F^j(\pi_F(\mathbf{x}_0)$ converges to the fixed point of T_F which is $\pi_F(\mathbf{f}) = \frac{1}{2}(\mathbf{e}_1^F + \mathbf{e}_4^F)$. The initial point of the sequence is $\pi_F(\mathbf{x}_0) = \mathbf{e}_0^F$. Thus the difference vectors $\pi_F(\mathbf{x}_j) - \pi_F(\mathbf{x}_0)$ converge for $j \to \infty$ to the diameter vector $\frac{1}{2}(\mathbf{e}_1^F + \mathbf{e}_4^F) - \mathbf{e}_0^F$ of the window V_1, and therefore $|\pi_F(\mathbf{x}_j) - \pi_F(\mathbf{x}_0)| \to L$. This finishes the proof that the decaonal tiling D is not of projection type.

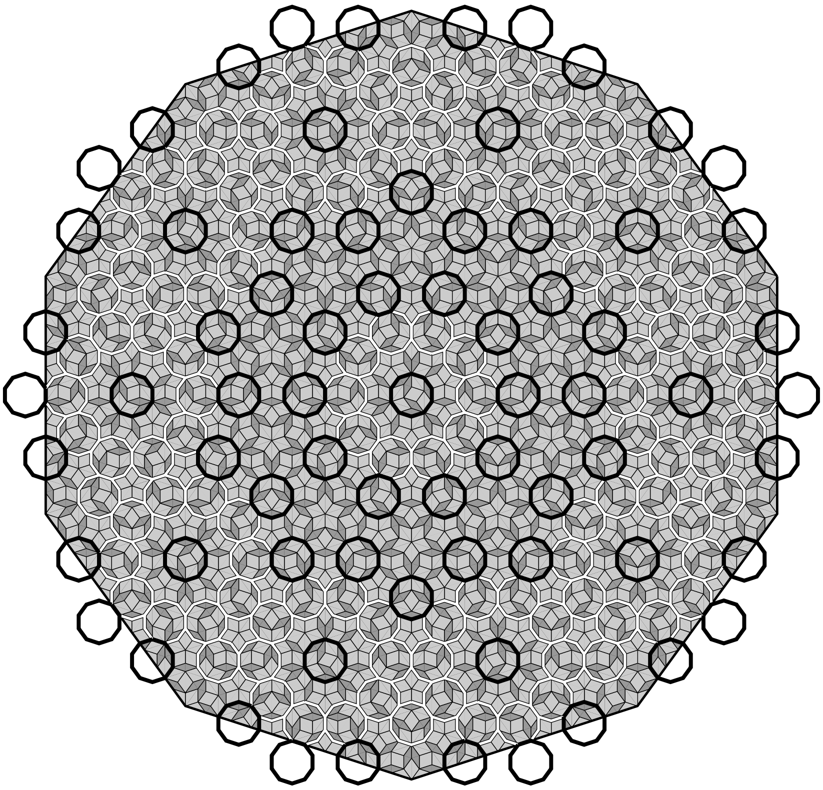
7. A link to traditional Islamic art

Some years ago, Peter Li and Paul Steinhard [4] (see also [8, 5]) observed that Penrose tilings are closely related to certain 17th century patterns in Islamic art. The subsequent figure shows an example from one of the entrance gates (called “The Master”) into the courtyard of the Friday Mosque at Isfahan, Iran. There is self similarity, and there are regular pentagons, like in Penrose tilings. However, the Isfahan pattern shows an exact decagonal symmetry which is impossible for Penrose tilings. Therefore the link between the two patterns is not completely obvious. It is only revealed when we consider the pattern of all Penrose decagons in a Penrose tiling. This can have a quasi-decagonal symmetry, and on our extended Penrose decagon tiling D, this symmetry becomes the more exact the farther we move away from the center. Thus we believe that the Penrose pattern D is the closest one to the Isfahan pattern.





The next figure shows the 8-fold subdivided Penrose decagon D_8. The Penrose decagons which correspond to the white circles in the Isfahan pattern have been marked dark. Other Penrose decagons which are marked white correspond to the above figure composed of black and white stones. The whole complicated arrangement is part of the geometry of the extended Penrose decagon tiling. Centuries ago it has been discovered by Iranian artists.



8. Discussion

Any finite Penrose tiling is of projection type. Infinite Penrose tilings arise as unions of infinitely many finite Penrose tilings P_j with $P_j \subset P_{j+1}$. It may happen that one of the common vertex points $p \in \bigcup_j P_j$ moves to the boundary of the window for P_j as $j \to \infty$. If this happens, then $P = \bigcup_j P_j$ cannot be a projection tiling. We have seen one example, the extended Penrose decagon tiling. However we expect that this phenomenon is not a rare exception but may happen quite often, maybe even in the generic case.

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