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COMMENT

Numerical calculation of thermal noise-voltage in a Josephson junction of finite capacitance

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In the paper by the late J. Kurkijärvi and myself (KA) [1] a molecular dynamics method is used to simulate the thermal agitation of the phase variable in a current carrying Josephson junction. Since the method is general and may be useful in other contexts, it is briefly described with some new details added.

Consider a classical particle of mass M, position X, and momentum P performing a one-dimensional thermal Brownian motion in a potential U. The process may be described by the Langevin equations

$$\dot{X} = P/M\,;\ \dot{P} = -\frac{dU}{dx} - \eta P + L(t), \quad (1)$$

where dots indicate derivatives with respect to the time t, and the symbols not yet defined are η the average dissipation and $L(t)$ the fluctuating force. It is known that to lead to thermal equilibrium at temperature T (in energy units) the fluctuating force must have the autocorrelation function $\langle L(t)L(t')\rangle = 2\eta MT\delta(t-t')$, where the brackets indicate an appropriate average.

The basic idea of the method introduced in KA is to simulate the fluctuating force $L(t)$ by random impulses describing elastic collisions with an ideal gas of light particles of mass m, assumed to be always in equilibrium. From the collision dynamics one calculates that random impulses $2p_i$ drawn from a set distributed according to $g(p) = (|p|/2mT)\exp(-p^2/2mT)$, $-\infty < p < \infty$, occurring at random time intervals t_i from a set distributed according to $f(t) = \nu\exp(-\nu t)$, $0 < t < \infty$, are required. With a given choice of $m \ll M$, ν must be chosen to satisfy $\nu = \eta M/4m$, this being the relation between the mean frequency of impact ν and the damping constant η.

Although the algorithm outlined in the last paragraph follows so directly from the physically motivated model that it cannot be wrong, it is worth verifying that the autocorrelation function of $L(t)$ so produced is indeed correct.

In the model

$$L(t) = \sum_{i=1}^{\infty} 2p_i\delta(t-T_i),\ T_i = \sum_{j\le i} t_j.$$

It follows that

$$\begin{aligned}&\langle L(t)L(t')\rangle = \langle 4p_1^2\rangle\\ &\times\big\langle\delta(t-t_1)+\delta(t-t_1-t_2)\\ &+\delta(t-t_1-t_2-t_3)\\ &+\delta(t-t_1-t_2-t_3-t_4)+\ldots\big\rangle\\ &\times\delta(t-t'). \end{aligned} \quad (2)$$

The fact that $\langle p_i p_j\rangle = 0$, for $i \ne j$, has been used to keep only diagonal terms in a double sum. In Eq. (2), the first term (the average squared impulse) is independent of its index and of time: it has been taken out of the time-average. Its value from the given distribution is $8mT$.

The average of the sum of δ-functions is ν. This can be seen both intuitively and formally. The intuitive argument is that, since the mean time interval between the δ-functions is ν^{-1}, the integral of the average over a time N/ν is N.

The average can also be done exactly term by term. The first term is simply

$$\begin{aligned}&\langle\delta(t-t_1)\rangle\\ &= \int_0^\infty dt_1\ \nu\,\mathrm{e}^{-\nu t_1}\delta(t-t_1) = \nu\,\mathrm{e}^{-\nu t}.\end{aligned} \quad (3)$$

For the last term shown the average is

$$\begin{aligned}&A_4 \equiv \langle\delta(t-t_1-t_2-t_3-t_4)\rangle\\ &= \int_0^\infty dt_2\int_0^\infty dt_3\int_0^\infty\\ &\times dt_4\ \nu^4\mathrm{e}^{-\nu t}\Theta(t-t_2-t_3-t_4)\\ &= \int_0^t dt_2\int_0^{t-t_2} dt_3\int_0^{t-t_2-t_3}\\ &\times dt_4\ \nu^4\mathrm{e}^{-\nu t}\\ &= \frac{1}{3!}\nu^4\,t^3\,\mathrm{e}^{-\nu t}.\end{aligned} \quad (4)$$

Above, the first equality comes from using the δ-function to collapse the t_1 integral, the step function Θ imposing the requirement that t_1 be positive.

[The average A_4 may also be evaluated using Fourier transforms and a contour integration:

$$A_4 = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left[\frac{\nu}{\omega + i\nu}\right]^4$$

$$= -i\nu^4 \frac{1}{3!} \frac{d^3}{d\omega^3} e^{-i\omega t}|_{\omega=-i\nu}$$

$$= \nu \frac{1}{3!} (\nu t)^3 e^{-\nu t}.] \tag{5}$$

As illustrated, the successive terms in the sum of δ-functions, when averaged over the random time intervals, produce an infinite series for $\exp(\nu t)$ which exactly compensates for and removes the decaying exponential in Eq. (3)! The intuitive argument that the average of the sum of δ-functions is equal to ν is thus formally verified.

Inserting these evaluations into Eq. (2) one obtains

$$\langle L(t)L(t')\rangle = 8mT\nu\delta(t-t')$$

$$= 2\eta MT\delta(t-t'). \tag{6}$$

Note that in the final result the mass m of the bath particles no longer appears, but that the temperature of the bath has been communicated to the Brownian particle.

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References

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