Lecture Notes on Symmetric Spaces

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0. Introduction

Riemannian symmetric spaces are the most beautiful and most important Riemannian manifolds. On the one hand, this class of spaces contains many prominent examples which are of great importance for various branches of mathematics, like compact Lie groups, Grassmannians and bounded symmetric domains. Any symmetric space has its own special geometry; euclidean, elliptic and hyperbolic geometry are only the very first examples. On the other hand, these spaces have much in common, and there exists a rich theory. The purpose of these notes is to give a brief introduction to the theory and to some of the examples.

Symmetric spaces can be considered from many different points of view. They can be viewed as Riemannian manifolds with point reflections or with parallel curvature tensor or with special holonomy or as a homogeneous space with a special isotropy or special Killing vector fields, or as Lie triple systems, or as a Lie group with a certain involution. We are trying to discuss all these aspects. Therefore, the chapters are only loosely connected and can be read almost separately.

Prerequisites are the fundamental concepts of Riemannian geometry and some basic knowledge of Lie groups which can be found e.g. in Chapter 4 of Cheeger-Ebin [CE]. The main reference is Helgason [H], but also Loos [L] and Besse [B]. Recently, P. Eberlein [E] gave a beautiful approach to symmetric spaces of noncompact type. We owe thanks also to Hermann Karcher for his lectures on symmetric spaces.

1. Definition and Examples

A (Riemannian) symmetric space is a Riemannian manifold S with the property that the geodesic reflection at any point is an isometry of S. In other words, for any $x \in S$ there is some $s_x \in G = I(S)$ (the isometry group of S) with the properties

$$s_x(x) = x, \quad (ds_x)_x = -I. \tag{$*$}$$

This isometry s_x is called *symmetry* at x. As a first consequence of this definition, S is geodesically complete: If a geodesic γ is defined on $[0, s)$, we may reflect it by $s_{\gamma(t)}$ for some $t \in (s/2, s)$, hence we may extend it beyond s. Moreover, S is homogeneous, i.e. for any two points $p, q \in M$ there is an isometry which maps p onto q. In fact, if we connect p and q by a geodesic segment γ (which is possible since S is complete) and let $m \in \gamma$ be its mid point, then $s_m(p) = q$. Thus G acts transitively. Let us fix a base point $p \in S$. The closed subgroup $G_p = \{g \in G;\ g(p) = p\}$ is called the *isotropy group* and will be denoted by K. The differential at p of any $k \in K$ is an orthogonal transformation of T_pS. Recall that the isometry k is determined by its differential dk_p; thus we may view K also as a closed subgroup of $O(T_pS)$ (the orthogonal group on T_pS) using this embedding $k \mapsto dk_p$ which is called *isotropy representation*. In particular, K is compact.

Vice versa, if S is any homogeneous space, i.e. its isometry group G acts transitively, then S is symmetric if and only if there exists a symmetry s_p (an isometry satisfying $(*)$) for some $p \in S$. Namely, the symmetry at any other point $q = gp$ is just the conjugate $s_q = gs_pg^{-1}$. Thus we have seen:

Theorem 1. *A symmetric space S is precisely a homogeneous space with a symmetry s_p at some point $p \in S$.* □

As usual, we may identify the homogeneous space S with the coset space G/K using the G-equivariant diffeomorphism $gK \mapsto gp$. In particular, $\dim S = \dim G - \dim K$.

Example 1: Euclidean Space. Let $S = \mathbb{R}^n$ with the euclidean metric. The symmetry at any point $x \in \mathbb{R}^n$ is the point reflection $s_x(x + v) = x - v$. The isometry group is the euclidean group $E(n)$ generated by translations and orthogonal linear maps; the isotropy group of the origin O is the orthogonal group $O(n)$. Note that the symmetries do not generate the full isometry group $E(n)$ but only a subgroup which is an order-two extension of the translation group.

Example 2: The Sphere. Let $S = S^n$ be the unit sphere in $\mathbb{R}^{n+1}$ with the standard scalar product. The symmetry at any $x \in S^n$ is the reflection at the line $\mathbb{R}x$ in $\mathbb{R}^{n+1}$, i.e. $s_x(y) = -y + 2\langle y, x\rangle x$ (the component of y in x-direction, $\langle y, x\rangle x$, is preserved while the orthogonal complement $y - \langle y, x\rangle x$ changes sign). In this case, the symmetries generate the full isometry group which is the orthogonal group $O(n + 1)$. The isotropy group of the last standard unit vector $e_{n+1} = (0, ..., 0, 1)^T$ is $O(n) \subset O(n + 1)$.

Example 3: The Hyperbolic Space. The hyperbolic space is defined quite similar, but instead of the standard scalar product on $\mathbb{R}^{n+1}$ we use the Lorentzian indefinite scalar product

$$(x, y) := \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}.$$

Then we let $S = H^n$ be one sheet of the two-sheeted hyperboloid $\{(x, x) = -1\}$, more precisely,

$$H^n = \{x \in \mathbb{R}^{n+1};\ (x, x) = -1,\ x_{n+1} > 0\}.$$

The induced scalar product on $T_x H^n = \{v \in \mathbb{R}^{n+1};\ (x, v) = 0\}$ is positive definite and turns H^n into a Riemannian manifold. As before, for any $x \in H^n$, the restriction to H^n of the Lorentzian reflection $s_x(y) = -y + 2(y, x)x$ is the symmetry at x (this time, the component of y in x-direction is $-(y, x)x$), and all the symmetries generate the full isometry group which is the group of "future preserving" Lorentz transformations $O(n, 1)^+$; "future preserving" means that H^n is not mapped to $-H^n$ but preserved. The isotropy group of e_{n+1} is again the subgroup $O(n) \subset O(n, 1)^+$. More precisely, H^n is called the *real* hyperbolic space. There exist also complex and quaternionic hyperbolic spaces and a 16 dimensional hyperbolic "plane" over the octonions.

Example 4: The Orthogonal Group. Let $S = O(n) = \{g \in \mathbb{R}^{n \times n};\ g^T g = I\}$. This is a submanifold of the matrix space $\mathbb{R}^{n \times n}$ since I is a regular value of the smooth map $x \mapsto x^T x :\ \mathbb{R}^{n \times n} \to S(n)$ (where $S(n)$ denotes the space of symmetric matrices). The Riemannian metric on $O(n)$ is induced from the trace scalar product on $\mathbb{R}^{n \times n}$:

$$\langle x, y \rangle := \operatorname{trace} x^T y = \sum_{ij} x_{ij} y_{ij}.$$

Note that left and right multiplications with orthogonal matrices preserve this inner product: For any $g \in O(n)$ and $x, y \in \mathbb{R}^{n \times n}$ we have

$$\begin{aligned}
\langle gx, gy \rangle &= \operatorname{trace}(x^T g^T g y) = \operatorname{trace} x^T y = \langle x, y \rangle, \\
\langle xg, yg \rangle &= \operatorname{trace}(g^T x^T y g) = \operatorname{trace}(g^{-1} x^T y g) = \operatorname{trace} x^T y = \langle x, y \rangle.
\end{aligned}$$

Further, right and left multiplications with $g \in O(n)$ preserve the subset $O(n) \subset \mathbb{R}^{n \times n}$, so they act as isometries on $O(n)$ turning $O(n)$ into a homogeneous space. Moreover, consider the linear map s_I on $\mathbb{R}^{n \times n}$ given by $s_I(x) = x^T$ which is also an isometry of $O(n)$ since it preserves $O(n) \subset \mathbb{R}^{n \times n}$ and the scalar product on $\mathbb{R}^{n \times n}$. This is the symmetry at the identity matrix $I \in O(n)$ since it fixes I and acts as $-I$ on the tangent space $T_I O(n) = \{v \in \mathbb{R}^{n \times n};\ v^T = -v\}$. Consequently, the symmetry at an arbitrary element $g \in O(n)$ is given by $s_g(x) = g(s_I(g^T x)) = g x^T g$. (This may be checked directly: In fact $s_g(g) = g$ and for any $gv \in T_g O(n) = g T_I O(n)$ we have $ds_g(gv) = s_g(gv) = gv^T = -gv$.)

Quite similar arguments apply to the unitary group $U(n) = O(2n) \cap \mathbb{C}^{n \times n}$ and the symplectic unitary group $Sp(n) = O(4n) \cap \mathbb{H}^{n \times n}$; we only have to replace x^T by $x^* = \bar{x}^T$.

Example 5: Compact Lie groups. More generally, let $S = G$ be a compact Lie group with biinvariant Riemannian metric, i.e. left and right translations $L_g, R_g : G \to G$ act as isometries for any $g \in G$. Then G is a symmetric space where the symmetry at the unit element $e \in G$ is the inversion $s_e(g) = g^{-1}$. Then $s_e(e) = e$ and $ds_e v = -v$ for any $v \in \mathfrak{g} = T_e G$, so $(*)$ is satisfied. We have to check that s_e is an isometry, i.e. $(ds_e)_g$ preserves the metric for any $g \in G$. This is certainly true for $g = e$, and for arbitrary $g \in G$ we have the relation $s_e \circ L_g = R_{g^{-1}} \circ s_e$ which shows

$$(ds_e)_g \circ (dL_g)_e = (dR_{g^{-1}})_e \circ (ds_e)_e.$$

Thus $(ds_e)_g$ preserves the metric since so do the other three maps in the above relation.

Example 6a: Projection model of the Grassmannians. Let $S = G_k(\mathbb{R}^n)$ be the set of all k-dimensional linear subspaces of $\mathbb{R}^n$ ("*Grassmann manifold*"). The group $O(n)$ acts transitively on this set, and the isotropy group of the standard $\mathbb{R}^k \subset \mathbb{R}^n$ is $O(k) \times O(n-k) \subset O(n)$. The symmetry s_E at any $E \in G_k(\mathbb{R}^n)$ will be the reflection s_E with fixed space E, i.e. with eigenvalue 1 on E and -1 on $E^\perp$.

But what is the manifold structure and the Riemannian metric on $G_k(\mathbb{R}^n)$? One way to see this is to embed $G_k(\mathbb{R}^n)$ into the space $S(n)$ of symmetric real $n \times n$ matrices: We assign to each k-dimensional subspace $E \subset \mathbb{R}^n$ the orthogonal projection matrix p_E with eigenvalues 1 on E and 0 on $E^\perp$. Let

$$P(n) = \{p \in S(n);\ p^2 = p\}$$

denote the set of all orthogonal projections. This set has several mutually disconnected subsets, corresponding to the trace of the elements which here is the same as the rank:

$$P(n)_k = P(n) \cap S(n)_k, \quad S(n)_k = \{x \in S(n);\ \operatorname{trace} x = k\}.$$

Now we may identify $G_k(\mathbb{R}^n)$ with $P(n)_k \subset S(n)$, using the embedding $E \mapsto p_E$ which is equivariant in the sense

$$g p_E g^T = p_{gE}$$

for any $g \in O(n)$. In fact, each p_E lies in this set, and vice versa, a symmetric matrix p satisfying $p^2 = p$ has only eigenvalues 1 and 0 with eigenspaces $E = \operatorname{im} p$ and $E^\perp = \ker p$, hence $p = p_E$, and the trace condition says that E has dimension k. Unfortunately, 0 is not a regular value of the map $F : S(n) \to S(n)$, $F(p) = p^2 - p$, defining the subset $P(n)_k \subset S(n)_k$, hence $P(n)_k$ is not a regular level set. However, it is still a submanifold of the affine space $S(n)_k$ since it is the conjugacy class of the matrix

$$p_0 = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

i.e. the orbit of p_0 under the action of the group $O(n)$ on $S(n)$ by conjugation. The isotropy group of p_0 is $O(k) \times O(n-k) \subset O(n)$. A complement of $T_I(O(k) \times O(n-k))$ in $T_I O(n)$ is the space of matrices of the type

$$\begin{pmatrix} 0 & -L^T \\ L & 0 \end{pmatrix}$$

with arbitrary $L \in \mathbb{R}^{(n-k)\times k}$, thus $P(n)_k = G_k(\mathbb{R}^n)$ has dimension $k(n-k)$. Since $G_k(\mathbb{R}^n) \subset F^{-1}(0)$, the kernel $\ker dF_p$ is contained in $T_p(G_k(\mathbb{R}^n))$. But the subspace $\ker dF_p = \{v \in S(n);\ vp + pv = v\}$ is isomorphic to $\operatorname{Hom}(E, E^\perp)$ since it contains precisely the symmetric matrices mapping $E = \operatorname{im} p$ into $E^\perp$ and vice versa (write the equation $vp + pv = v$ as $vp = (I - p)v$ and recall that $I - p = p_{E^\perp}$). Thus $\ker dF_p$ has dimension $k(n-k)$ and thus exhausts the tangent space $T_p G_k(\mathbb{R}^n)$.

Now we equip $P(n)_k \subset S(n)$ with the metric induced from the trace scalar product $\langle x, y\rangle = \operatorname{trace}(x^T y) = \operatorname{trace}(xy)$ on $S(n)$. The group $O(n)$ acts isometrically on $S(n)$ by conjugation and preserves $P(n)_k$, hence it acts isometrically on $P(n)_k$. In particular, let $s_E \in O(n)$ be the reflection at the subspace E and let $\hat{s}_E$ be the corresponding conjugation, $\hat{s}_E(x) = s_E x s_E$. This is an isometry fixing p_E, and since s_E fixes E and reflects $E^\perp$, the conjugation $\hat{s}_E$ maps any $x \in T_pG_k(\mathbb{R}^n)$ into $-x$ (recall that x is a linear map from E to $E^\perp$ and vice versa). Thus $\hat{s}_E$ is the symmetry at p_E.

The Grassmannians $G_k(\mathbb{C}^n)$ and $G_k(\mathbb{H}^n)$ are defined and embedded analogously where $S(n)$ has to be replaced by the space of hermitean matrices over $\mathbb{C}$ and $\mathbb{H}$. A case of particular interest is $k = 1$; these are the projective spaces over $\mathbb{R}, \mathbb{C}, \mathbb{H}$. When $n = 2$, $k = 1$, there is even an analogous construction for the octonions which yields the octonionic projective plane (e.g. cf. [Hi]).

Example 6b: Reflection model of the Grassmannians. Instead of identifying a subspace $E \in G_k(\mathbb{R}^n)$ with its orthogonal *projection* $p = p_E$ which has eigenvalues 1 on E and 0 on $E^\perp$, we may as well assign to E the *reflection* $s = s_E$ at the space E, i.e. the matrix with eigenvalues 1 on E and -1 on $E^\perp$. Clearly we have

$$s_E + I = 2p_E$$

and therefore $2p_E$ and s_E just differ by the identity matrix which commutes with conjugations, hence their conjugacy classes are parallel, separated by the constant matrix I. So the two models are very much alike, and in particular, their tangent and normal spaces are exactly the same. However, the new model ("*reflection model*") has an extra feature: Reflections belong to the symmetric matrices as well as to the orthogonal ones. In fact, the set $R(n)$ of all reflections is precisely the intersection of these two sets,

$$R(n) = O(n) \cap S(n) = \{s \in O(n);\ s^T = s\},$$

since $s \in O(n) \cap S(n) \iff s^{-1} = s^T = s$, hence $s^{-1} = s$ which is the reflection property. In fact, $R(n) \subset O(n)$ is the fixed point set of the linear map $\tau : x \mapsto x^T$ on $\mathbb{R}^{n\times n}$ which is an isometry with respect to the trace scalar product $\langle x, y\rangle = \operatorname{trace} x^T y$ and preserves $O(n)$, see Example 4. The fixed set of an isometry on a Riemannian manifold is always a disjoint union of totally geodesic submanifolds, the connected components. In the case at hand, these are labeled by the dimension k of the fixed space of the element or equivalently by the value of the trace which is $k-(n-k) = 2k-n$:

$$R(n)_k = \{s \in R(n);\ \operatorname{trace} s = 2k - n\}$$

This is the set of all reflections whose fixed space is k-dimensional. Hence the map $G_k(\mathbb{R}^n) \to R(n)_k : E \mapsto s_E$ is bijective and $O(n)$-equivariant as before. Moreover, being a component of the fixed point set of an isometry, $R(n)_k \subset O(n)$ is totally geodesic and hence again a symmetric space by general theory (see Theorem 8 below); in fact the symmetry of the subspace is the restriction of the symmetry of the large space $O(n)$. From Example 4 we know that the symmetry at $g \in O(n)$ is the map $x \mapsto gx^Tg$; for $g = s \in R(n)_k$ and $x \in S(n)$ this map becomes the conjugation by s. The tangent space is $T_sR(n)_k = T_sO(n) \cap S(n) = sA(n) \cap S(n)$. Hence any $x \in T_sR(n)_k$ has the form $x = sa$ for some $a \in A(n)$. Since $(sa)^T = -as$, we have $sa \in S(n) \iff sa = -as \iff a$ interchanges the two eigenspaces $E, E^\perp$ of s, and so does $x = sa$. Counting dimensions we see $T_sR(n) \cong \operatorname{Hom}(E, E^\perp)$ as above.

Example 7: Complex Structures on $\mathbb{R}^n$. Let S be the set of orthogonal complex structures in $\mathbb{R}^n$ for even $n = 2m$. The elements of S are orthogonal $n \times n$-matrices j with $j^2 = -I$ or $j^{-1} = -j$. From orthogonality we also have $j^{-1} = j^T$, and thus we get a third relation $j^T = -j$, i.e. j lies in the set $A(n)$ of antisymmetric matrices (which is the tangent space of $O(n)$ at I). Any two of the three relations

$$j^{-1} = j^T, \;\; j^{-1} = -j, \;\; j^T = -j$$

imply the third one. Thus

$$S = O(n) \cap A(n) = \{j \in A(n);\; j^2 = -I\}.$$

Moreover, since any $j \in S$ is an orthogonal matrix with eigenvalues $+i, -i$ of multiplicity m, all j's are conjugate and S is a conjugacy class, an orbit under the action of $O(n)$ by conjugation. The isotropy group of the standard complex structure i of $\mathbb{R}^{2m} = \mathbb{C}^m$ is the unitary group $U(m) \subset O(2m)$.*) Moreover, it is again a fixed point set component of an isometry, the map $x \mapsto -x^T$ on $\mathbb{R}^{n\times n}$ which preserves the submanifold $O(n)$. Though S is again not a regular preimage (the differential of the defining map $F : j \mapsto j^2 + I$ is certainly not onto, by reasons of dimension), we have $T_jS = \ker\, dF_j = \{v \in A(n);\; vj + jv = 0\}$; in fact, $\ker\, dF_j$ is the space of elements of $A(n)$ which anticommute with j; this is complementary to the subspace $T_IU(m) \subset A(n)$ containing the elements commuting with j, so it has the right dimension. Now the symmetry s_j is just the conjugation with j: it fixes j and maps $v \in T_jS$ to $-v$ since $jvj^{-1} = -vjj^{-1} = -v$.

The space of quaternionic structures on $\mathbb{C}^{2m}$ is treated similarly; it is embedded into the space of antihermitean complex $n \times n$-matrices and is an orbit of the group $U(2m)$ acting by conjugation; the isotropy group is $Sp(m) \subset U(2m)$.

Example 8: Real structures on $\mathbb{C}^n$. Let S be the set of real structures on $\mathbb{C}^n$. A *real structure* on $\mathbb{C}^n = \mathbb{R}^{2n}$ is a reflection κ at a totally real subspace E of half dimension where "totally real" means $iE \perp E$. In other words, κ is a reflection which is *complex antilinear*, i.e. it anticommutes with the complex structure i. In general, if a reflection anticommutes with i then i interchanges its $+1$ and -1 eigenspaces E_+ and E_-, so E_+ is totally real, and vice versa, a reflection with totally real $+1$ eigenspace anticommutes with i. Since any reflection is symmetric, we may consider S as a subset of $S(2n)_-$ which by definition is the intersection of $S(2n)$ with the space of complex antilinear maps on $\mathbb{C}^n$. This is a totally geodesic subspace of $R(2m)_m$ (see Example 6b), in fact a fixed point component of the isometric linear map $x \mapsto jxj$ of $\mathbb{R}^{n\times n}$ which preserves $R(2m)_m$:

$$S = S(2n)_- \cap O(2n) = \{\kappa \in R(2m)_n;\; j\kappa j = \kappa\}.$$

An orthonormal basis of E is a unitary basis of $\mathbb{C}^n$ and vice versa, the real span of a unitary basis is an n-dimensional totally real subspace. Thus $U(n)$ acts transitively on S. More precisely, S is the orbit of the standard complex conjugation $\kappa_0(v) = \bar{v}$ in $\mathbb{C}^n$ (which is the reflection at the standard $\mathbb{R}^n \subset \mathbb{C}^n$) under the conjugacy action of $U(n)$. The isotropy group of κ_0 is $O(n) \subset U(n)$, hence $S \cong U(n)/O(n)$. For the map $F(x) = x^Tx - I$ which defines $S \subset S(2n)_-$ we have

$$\ker\, dF_\kappa = \{v \in S(2n)_-;\; v\kappa + \kappa v = 0\}$$

*) In particular, $S \cong O(2m)/U(m)$ has two connected components like $O(2m)$ since $U(m)$ is connected. Unlike the situation in the previous example, the conjugacy class of j with respect to $SO(n)$ is strictly smaller than the $O(n)$-conjugacy class.

Thus $v \in \ker dF_\kappa$ iff the $\mathbb{C}$-linear map κv is antisymmetric as a real matrix ($(\kappa v)^T = v\kappa = -\kappa v$), hence $\kappa v \in T_I U(n) = T_I O(2n) \cap \mathbb{C}^{n\times n}$. Moreover, κv anticommutes with κ, so it is purely imaginary with respect to the real structure κ. On the other hand, the purely imaginary matrices in $T_I U(n)$ form a complement to $T_I O(n)$, thus $\ker dF_\kappa = T_\kappa S$ by reasons of dimension. Now it is easy to see that the symmetry s_κ is given by the conjugation with κ, i.e. $s_\kappa(x) = \kappa x \kappa$: it fixes κ and acts as $-I$ on $T_\kappa S$.

The space of complex structures on $\mathbb{H}^n$ is treated similarly; as a quotient space it is $Sp(n)/U(n)$.

Example 9: Positive Definite Matrices. Let $S = P(n)$ be the set of positive definite real symmetric $n \times n$-matrices which is an open subset of the vector space $S(n)$ of all symmetric matrices. We define the following Riemannian metric on $P(n)$: For any $v, w \in T_p P(n) = S(n)$ we put

$$\langle v, w\rangle_p = \operatorname{trace} vp^{-1}wp^{-1} = \operatorname{trace} p^{-1}vp^{-1}w.$$

The group $G = GL(n, \mathbb{R})$ acts on $P(n)$ by $g(p) := gpg^T$, and this action is isometric with respect to the chosen metric: For any $x \in S(n) = T_p P(n)$ we have $dg_p x = gxg^T$ and hence

$$\begin{aligned} \langle dg_p v, dg_p w\rangle_{g(p)} &= \operatorname{trace} gvg^T(gpg^T)^{-1}gwg^T(gpg^T)^{-1} \\ &= \operatorname{trace} gvp^{-1}wp^{-1}g^{-1} \\ &= \langle v, w\rangle_p \end{aligned}$$

for all $v, w \in T_p P(n)$. Since any $p \in P(n)$ can be written as $p = g^T g = g(I)$ for some $g \in G$, this action is transitive, and the isotropy group of the identity matrix $I \in P(n)$ is $O(n)$. Further, the inversion $s_I(p) = p^{-1}$ is also an isometry of $P(n)$: In fact, since $(ds_I)_p x = -p^{-1}xp^{-1}$, we have

$$\begin{aligned} \langle (ds_I)_p v, (ds_I)_p w\rangle_{p^{-1}} &= \operatorname{trace} p^{-1}vp^{-1}pp^{-1}wp^{-1}p \\ &= \operatorname{trace} p^{-1}vp^{-1}w \\ &= \langle v, w\rangle_p \end{aligned}$$

for all $v, w \in T_p P(n)$. Since s_I fixes I and acts as $-I$ on $T_I P(n)$, it is the symmetry at I. The symmetry at an arbitrary $p \in P(n)$ is $s_p(q) = pq^{-1}p$.

Remark Examples 2,4,6,7,8 arise as so called *extrinsic symmetric spaces*: A submanifold $S \subset \mathbb{R}^N$ is called extrinsic symmetric if it is preserved by the reflections at all of its normal spaces. More precisely, let s_p be the isometry of $\mathbb{R}^N$ fixing p whose linear part ds_p acts as identity I on the normal space $\nu_p S$ and as $-I$ on the tangent space $T_p S$; then S is extrinsic symmetric if $s_p(S) = S$ for all $p \in S$. Extrinsic symmetric spaces (also called *symmetric R-spaces*) are classified (cf. [KN], [F], [EH]).

2. Transvections and Holonomy

We saw in Section 1 that the symmetry at the mid point maps any point $p \in S$ to any other point $q \in S$. But we shall find another isometry with even better properties. Let γ be the geodesic segment connecting p and q such that $\gamma(0) = m$ is the mid point, and extend it to a complete geodesic. The symmetry s_m reflects each parallel vector field X along γ; in fact, being an isometry it maps X along γ onto another parallel vector field $\tilde{X}$ along $\tilde{\gamma}$ with $\tilde{\gamma}(t) = s_p(\gamma(t)) = \gamma(-t)$ and $ds_m.X(0) = -X(0)$. Thus

$$ds_m.X(t) = -X(-t)$$

for all $t \in \mathbb{R}$. Now the composition of two symmetries, say $\tau = s_q \circ s_m$, reflects X twice, hence it keeps X invariant. More precisely, if $q = \gamma(s)$,

$$\tau(\gamma(t)) = \gamma(t+s), \quad d\tau_{\gamma(t)}X(t) = X(t+s) \tag{$*$}$$

for any parallel vector field X along γ. Such isometry τ is called a *transvection* along γ. Since any isometry is determined by its value and its derivative at a single point, $(*)$ shows also that the transvections $\tau = \tau_s$ for variable $s \in \mathbb{R}$ form a one-parameter subgroup of G, i.e. $\tau_{s+s'} = \tau_s \circ \tau_{s'}$. In fact, by $(*)$, $\tau_s \circ \tau_{s'}$ is the isometry sending the point $\gamma(0)$ to $\gamma(s+s')$ and any vector $X \in T_{\gamma(0)}S$ to its parallel translate $P_{s+s'}X$ (where P_t denotes the parallel transport from $T_{\gamma(0)}S$ to $T_{\gamma(t)}S$ along γ), and the same holds for $\tau_{s+s'}$, hence $\tau_s \circ \tau_{s'} = \tau_{s+s'}$. More generally, for any two points $p, q \in S$, the composition $s_p \circ s_q$ is a transvection along any geodesic connecting p and q. Now we have proved a theorem with many strong consequences:

Theorem 2. *Each complete geodesic $\gamma : \mathbb{R} \to S$ is the orbit of a one-parameter group of isometries, the transvections along γ, which act as parallel transports along γ.* $\square$

Corollary 1. *For any $p \in S$, the holonomy group* Hol_p *is contained in the isotropy group $K = G_p$*

Proof Recall that the holonomy group Hol_p is the group of parallel translations along all closed curves starting and ending at p. We may approximate such a curve by a closed geodesic polygon. The parallel transport along any edge of the polygon is given by applying a transvection along that edge, and so the parallel transport along the full polygon is a composition of isometries which sends p back to itself, hence it is an element of the isotropy group $K = G_p$ (acting on T_pS by the isotropy representation). Since K is compact, the sequence of parallel transports along geodesic polygons approximating the given loop better and better has a convergent subsequence, hence $\mathrm{Hol}_p \subset K$.*) $\square$

Corollary 2. *The fundamental group of S is abelian.*

Proof Fix some $p \in S$ and consider the fundamental group $\pi = \pi_1(S,p)$ consisting of homotopy classes $[c]$ of loops $c : [0,1] \to S$ starting and ending at p while the group operation is by concatenation of loops. We will show that the inversion map $j : \alpha \mapsto \alpha^{-1}$ on π is a group homomorphism (and an anti-homomorphism anyway); this will show that π is abelian. Any smooth map $f : S \to S$ fixing p creates a homomorphism $f_* : \pi \to \pi$, $f_*([c]) = [f \circ c]$. We will use $f = s_p$ (the symmetry at p) and show that

*) In any Riemannian manifold we have kind of opposite relation: the isotropy group is always contained in the normalizer of the holonomy group. This is because isometries preserve parallel transport, hence for any $h = P_c \in \mathrm{Hol}_p$ (where c is a loop c starting and ending at p) and any isometry g fixing p we have $gP_cg^{-1} = P_{gc} \in \mathrm{Hol}_p$. Thus for a symmetric space G/K we have $\mathrm{Hol}_p \subset K \subset N(\mathrm{Hol}_p)$. We will see later (Theorem 7.2) that Hol_p and K have the same Lie algebra.

$(s_p)_* = j$. In fact, if $[c]$ is a nontrivial homotopy class, it contains a geodesic loop γ, i.e. $[c] = [\gamma|_{(0,1]}]$. Now γ is the orbit of a one-parameter group of isometries, and in particular, if $\gamma(1) = \gamma(0) = p$, we have $\gamma'(1) = \gamma'(0)$ (the tangent vector field is a restriction of the Killing field generated by the one-parameter group, see Section 3 below). Thus the geodesic loop is a closed (1-periodic) geodesic. Hence $s_p \circ \gamma$ given by $s_p(\gamma(t)) = \gamma(-t)$ is the same loop, parametrized in reverse direction, which implies $(s_p)_*[\gamma] = [\gamma]^{-1} = j([\gamma])$ and hence $j = (s_p)_*$ is a group homomorphism. $\square$

Corollary 3. *If S is simply connected and indecomposable (i.e. S does not split as a nontrivial Riemannian product), then S is isotropy irreducible, i.e. K acts irreducibly on T_pS. In particular, S is Einstein.*

Proof If the isotropy action of K on T_pS is reducible, i.e. T_pS splits as an orthogonal direct sum of nontrivial K-invariant subspaces $T_pS = V_1 \oplus V_2$, then V_1 and V_2 are also invariant under $\mathrm{Hol}_p \subset K$. But then, by de Rham's theorem (cf. [KNo]) S is reducible. Since the Ricci tensor at the point p is K-invariant, all its eigenspaces are K-invariant, and if S is isotropy irreducible there can be only one eigenspace, i.e. S is Einstein. $\square$

From now on, we will often assume that S is isotropy irreducible. This is more general than the assumption "indecomposable"; e.g. it includes the case of euclidean space $\mathbb{R}^n = E(n)/O(n)$ where $E(n)$ is the euclidean group.

Remark It is well known for any Riemannian manifold S that the curvature endomorphisms $R(v, w)$ for any $v, w \in T_pS$ belong to the Lie algebra of the holonomy group which in our case is contained in the Lie algebra $\mathfrak{k}$ of K. If we consider the trilinear map $(x, y, z) \mapsto R(x, y)z$ as an algebraic structure (a so called *triple product*) on T_pS, then K is part of its automorphism group since isometries preserve the curvature tensor. The Lie algebra of the automorphism group of any algebraic structure is its derivation algebra; hence the elements $A \in \mathfrak{k}$ act on T_pS by derivations with respect to R, i.e.

$$A.R(x, y)z = R(A.x, y)z + R(x, A.y)z + R(x, y)A.z.$$

This holds in particular for $A = R(v, w) = R_{vw}$:

$$R_{vw}R(x, y)z = R(R_{vw}x, y)z + R(x, R_{vw}y)z + R(x, y)R_{vw}z. \qquad (L)$$

A curvature tensor R with this property is called *Lie triple (product)*. We will come back to this point in Chapter 4.

3. Killing Fields

Let S be a symmetric space and fix a base point $p \in S$. Let $\mathfrak{g}$ be the Lie algebra of the isometry group G of S viewed as the space of Killing vector fields. This has two distinguished subspaces $\mathfrak{k}$ and $\mathfrak{p}$ where

$$\mathfrak{k} = \{X \in \mathfrak{g};\ X_p = 0\}, \quad \mathfrak{p} = \{X \in \mathfrak{g};\ (\nabla X)_p = 0\}.$$

Obviously, $\mathfrak{k}$ is the Lie algebra of the isotropy group $K = G_p$. But $\mathfrak{p}$ also has a geometric meaning: it is the space of *infinitesimal transvections* at p. Recall that for any geodesic γ_v starting from p with tangent vector $v \in T_pS$ there is a one-parameter group of transvections $g_v(t)$ translating γ_v and all parallel vector fields along γ_v. The infinitesimal transvections at p are the corresponding Lie algebra elements or Killing fields $V := \frac{d}{dt} g_v(t)\big|_{t=0}$, for any $v \in T_pS$. We claim that those form the subspace $\mathfrak{p}$. In fact, if $w \in T_pS$, we choose a curve $p(s)$ with $p(0) = p$ and $p'(0) = w$. Then

$$\nabla_w V = \frac{D}{\partial s}\frac{\partial}{\partial t} g_t(p(s))\Big|_{s=t=0} = \frac{D}{\partial t}\frac{\partial}{\partial s} g_t(p(s))\Big|_{0,0} = 0$$

since $\frac{\partial}{\partial s} g_t(p(s))|_{s=0} = dg_t.w$ is a parallel vector field along γ_v. This shows that all infinitesimal transvections at p are in $\mathfrak{p}$. Since a Killing field (like an isometry) is determined by its value and its derivative at a single point, the dimension of $\mathfrak{p}$ is not larger than the dimension of T_pS, hence the infinitesimal transvections at p exhaust $\mathfrak{p}$. Since $\mathfrak{k}$ and $\mathfrak{p}$ have no common intersection, we have a direct vector space decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

by reasons of dimension. Clearly, $\mathfrak{k} \subset \mathfrak{g}$ is a subalgebra; in fact, the Killing field $[V, W] = \nabla_V W - \nabla_W V$ vanishes at p if so do V and W. In order to determine $[V, W]$ we only have to compute its derivative at p. First we have to recall a more general fact:

Lemma. *On any Riemannian manifold S, a Killing field X satisfies the following covariant differential equation*

$$\nabla^2_{A,B} X + R(X, A)B = 0 \tag{K}$$

for arbitrary vector fields A, B where

$$\nabla^2_{A,B} X := (\nabla_A(\nabla X))B = \nabla_A \nabla_B X - \nabla_{\nabla_A B} X.$$

Proof Let us denote the left hand side of (K) by $L(A, B)$. Since any Killing field is a Jacobi field when restricted to a geodesic γ (it is the variation field of the family of geodesics $\gamma_s = g_s \circ \gamma$ where g_s is the one-parameter group of isometries corresponding to the Killing field), we have $\nabla^2_{A,A} X + R(X, A)A = 0$ for any vector field A, hence $L(A, A) = 0$ for any A which shows that the symmetric part $L(A, B)+L(B, A)$ vanishes. But the antisymmetric part vanishes anyway by Bianchi identity since $\nabla^2_{A,B} - \nabla^2_{B,A} = R(A, B)$:

$$L(A, B) - L(B, A) = R(A, B)X + R(X, A)B - R(X, B)A = 0 \qquad \square$$

Now for any $X, Y \in \mathfrak{k}$ and $V, W \in \mathfrak{p}$ we have $[X, Y] \in \mathfrak{k}$, $[V, W] \in \mathfrak{k}$ and $[X, V] \in \mathfrak{p}$; to see this we note that $\nabla_X Y$ and $\nabla_V W$ vanish at p due to $X_p = 0$ and $(\nabla W)_p = 0$, hence $[X, Y], [V, W] \in \mathfrak{k}$. Moreover, by (K) we have for any $A \in T_pS$

$$\nabla_A[X, V] = \nabla_A \nabla_X V - \nabla_A \nabla_V X \overset{*}{=} -R(V, A)X + R(X, A)V = 0$$

at p where $\overset{*}{=}$ is due to (K) and $(\nabla V)_p = 0$, hence $[X, V] \in \mathfrak{p}$.

Furthermore we have for any $U \in \mathfrak{p}$ at the point p (where $\nabla V, \nabla W = 0$):

$$\begin{aligned}\nabla_U[V, W] &= \nabla_U \nabla_V W - \nabla_U \nabla_W V \\ &= -R(W, U)V + R(V, U)W = R(V, W)U\end{aligned}$$

On the other hand, at p we also have

$$\nabla_U[V, W] = [U, [V, W]]$$

since $\nabla_{[V,W]} U$ vanishes at p. Thus we have expressed the curvature tensor in terms of the Lie algebra structure:

Theorem 3. *Let S be a symmetric space and $p \in S$. Let $\mathfrak{k}$ be the set of Killing fields vanishing at p and $\mathfrak{p}$ the set of infinitesimal transvections at p, i.e. the Killing fields with vanishing covariant derivative at p. Then*

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \ \ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \ \ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{C}$$

Further, the map $\mathfrak{p} \to T_pS$, $V \mapsto V_p$ is a linear isomorphism, and for all $U, V, W \in \mathfrak{p}$ we have

$$(R(V,W)U)_p = ([U,[V,W]])_p\,. \qquad \square$$

4. Cartan Involution and Cartan Decomposition

Theorem 4.1

a) *Let G be a connected Lie group with an involution (order-2 automorphims) $\sigma : G \to G$ and a left invariant metric which is also right invariant under the closed subgroup*

$$\hat{K} = \mathrm{Fix}(\sigma) = \{g \in G;\ g^\sigma = g\}$$

Let K be a closed subgroup of G with

$$\hat{K}^o \subset K \subset \hat{K}$$

where $\hat{K}^o$ denotes the connected component (identity component) of $\hat{K}$. Then $S = G/K$ is a symmetric space where the metric is induced from the given metric on G.

b) *Every symmetric space S arises in this way.*

Proof We prove Part b) first. Let S be a symmetric space. We have seen in Theorem 1 that the isometry group $\hat{G} = I(S)$ acts transitively and contains a point reflection or geodesic symmetry s_p at some $p \in S$. The conjugation by s_p defines an automorphism σ of $\hat{G}$,

$$\sigma(g) = s_p g s_p^{-1} = s_p g s_p.$$

Note that $s_p^2 = s_p \circ s_p$ is the identity element $e \in \hat{G}$ since s_p^2 is an isometry which has the same value and derivative at p as the identity e. Thus σ is also an automorphism of order 2, an *involution* of $\hat{G}$.

In Theorem 2 we have seen that the identity component $G = \hat{G}^o$ acts transitively too, and since the identity component is preserved by any automorphism, σ is an involution of G as well.

Since s_p acts as $-I$ on T_pS, it commutes with the action of the isotropy group K at p. In other words, K lies in the fixed point set of σ in G. Vice versa, if $g \in \mathrm{Fix}(\sigma)$, it commutes with s_p and hence it preserves the subset $F_{s_p} = \{x \in S;\ s_p(x) = x\}$, the fixed point set of s_p in S. But p is an isolated point of F_{s_p} since no nonzero vector $v \in T_pS$ is fixed by ds_p (it is mapped to $-v$). Hence $g(p) = p$ if g can be connected to e in $\mathrm{Fix}(\sigma)$, i.e. if g lies in the connected component $\mathrm{Fix}(\sigma)^o$. So we have seen

$$\mathrm{Fix}(\sigma)^o \subset K \subset \mathrm{Fix}(\sigma). \tag{F}$$

The mapping

$$\pi : G \to S, \quad g \mapsto gp$$

is a *submersion**) with fibres $\pi^{-1}(gp) = \{gk;\ k \in K\} = gK$, thus S is equivariantly diffeomorphic to the coset space G/K. In particular, $\pi_* := d\pi_e : \mathfrak{g} = T_eG \to T_pS$ is a K-equivariant linear map,

$$\pi_* \operatorname{Ad}(k)X = k_*\pi_*X$$

for any $k \in K$ and $X \in \mathfrak{g}$, and π_* is onto with kernel $\mathfrak{k} = T_eK$. By (F) we know that $\mathfrak{k}$ is the fixed space ($(+1)$-eigenspace) of σ_*. A canonical complement is the (-1)-eigenspace which we call $\mathfrak{p}$. Using the equivariant isomorphism $\pi_*|_{\mathfrak{p}} : \mathfrak{p} \to T_pS$, we transplant the inner product on T_pS to $\mathfrak{p}$, and we extend it to an $\operatorname{Ad}(K)$-invariant metric on $\mathfrak{g}$ by choosing any $\operatorname{Ad}(K)$-invariant metric on $\mathfrak{k}$ and declaring $\mathfrak{k} \perp \mathfrak{p}$. This extends to a left invariant metric on G which is also right invariant with respect to K, and the submersion $\pi : G \to S$ is Riemannian, i.e. $d\pi_g$ is isometric on the horizontal subspace $\mathcal{H}_g = (T_g(gK))^\perp = (L_g)_*\mathfrak{p} \subset T_gG$.

Vice versa, let G, K, σ be as in the assumption of Part a). The metric on G induces a metric on the coset space $S := G/K$ which thus becomes a Riemannian homogeneous space. To see that it is symmetric, by Theorem 1 we only have to find the symmetry $s = s_p$ at the point $p = eK \in S$. Since $K \subset \operatorname{Fix}(\sigma)$, we have $\sigma(K) = K$ and hence $\sigma : G \to G$ descends to a diffeomorphism $s : G/K \to G/K$. Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

be the decomposition of $\mathfrak{g} = T_eG$ into the (± 1)-eigenspaces of σ_*. We have $s(p) = p$ and $ds_p = -I$ since

$$T_p(G/K) = \mathfrak{g}/\mathfrak{k} = \{X + \mathfrak{k};\ X \in \mathfrak{g}\} = \{X + \mathfrak{k};\ X \in \mathfrak{p}\}$$

and $ds_p(X + \mathfrak{k}) = \sigma_*(X) + \mathfrak{k} = -X + \mathfrak{k}$ for all $X \in \mathfrak{p}$. It remains to show that s is an isometry, i.e $ds_{gp} : T_{gK}S \to T_{\sigma(gK)}S$ preserves the inner product. This is clear for $g = e$ since $ds_p = -I$, and it follows for arbitrary $g \in G$ from the observation $s(ghK) = g^\sigma h^\sigma K = g^\sigma s(hK)$, hence

$$s \circ L_g = L_{g^\sigma} \circ s$$

for any $g \in G$, where $L_g : S \to S$, $L_g(hK) = ghK$. Differentiating this equality at $p = eK$ we obtain

$$ds_{gp} \circ (dL_g)_p = (dL_{g^\sigma})_p \circ ds_p.$$

All these linear maps are bijective and $(dL_g)_p$, $(dL_{g^\sigma})_p$, ds_p preserve the inner products, thus the same holds for ds_{gp} which shows that s is an isometry and hence $S = G/K$ is symmetric. □

E.g. for the sphere $S^n = SO(n+1)/SO(n)$ the isotropy group $K = SO(n)$ is a proper subgroup of $\operatorname{Fix}(\sigma) = O(n)$, while in the real projective space $\mathbb{R}P^n = SO(n+1)/O(n)$ the two groups agree. In both cases we have $\operatorname{Fix}(\sigma)^o = SO(n)$.

Passing to the Lie algebra level we see that the differential σ_* of σ is an order-two automorphism of the Lie algebra $\mathfrak{g}$. Hence we get a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ and $\mathfrak{p}$ are the eigenspaces of σ_* corresponding to the eigenvalues 1 and -1. From (F) we see that $\mathfrak{k}$ is the Lie algebra of the isotropy group K. Moreover, the other eigenspace $\mathfrak{p}$ is also $\operatorname{Ad}(K)$-invariant since $\operatorname{Ad}(K)$ commutes with σ_*. In fact, $\mathfrak{p}$ is the space of infinitesimal transvections at p since the conjugation by s_p reverses the one-parameter group of transvections along any geodesic γ starting at p.

*) i.e. π and $d\pi_g$ are onto for any $g \in G$

Lemma *A vector space decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of a Lie algebra $\mathfrak{g}$ is the eigenspace decomposition of an order-two automorphism σ_* of $\mathfrak{g}$ if and only if*

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k},\ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p},\ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}. \tag{C}$$

Proof If $E_\lambda \subset \mathfrak{g}$ denotes the eigenspace corresponding to the eigenvalue λ of an automorphism σ_*, then $[E_\lambda, E_\mu] \subset E_{\lambda \cdot \mu}$ since

$$\sigma_*[X_\lambda, X_\mu] = [\sigma_* X_\lambda, \sigma_* X_\mu] = \lambda\mu[X_\lambda, X_\mu].$$

For $\lambda, \mu \in \{1, -1\}$ this shows (C). Vice versa, if a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with (C) is given, then the linear map σ_* on $\mathfrak{g}$ which is I on $\mathfrak{k}$ and $-I$ on $\mathfrak{p}$ is a Lie algebra automorphism. □

A decomposition of a Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with (C) such that $\mathrm{ad}(\mathfrak{k})|_{\mathfrak{p}}$ is the Lie algebra of a compact subgroup of $GL(\mathfrak{p})$ will be called *Cartan decomposition* and the corresponding involution σ_* *Cartan involution*.

Now starting from a Lie algebra $\mathfrak{g}$ with Cartan involution σ_* we construct a symmetric space as follows. Recall that any automorphism of the Lie algebra $\mathfrak{g}$ is the differential of an automorphism of the corresponding simply connnected Lie group G. Hence we obtain an involution σ on G such that $\mathrm{Ad}(\mathrm{Fix}(\sigma))$ acts as a compact group on $\mathfrak{p}$. Let $K \subset G$ be any closed subgroup satisfying (F). Fix any $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{p}$ (which exists since $\mathrm{Ad}(K)$ acts on $\mathfrak{p}$ as a compact group). This determines a G-invariant metric on $S := G/K$, making it a symmetric space by Theorem 4.1. Thus we have shown the following

Theorem 4.2 *Any symmetric space S determines a Cartan decomposition on the Lie algebra of Killing fields. Vice versa, to any Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ there exists a unique simply connected symmetric space $S = G/K$ where G is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and K the connected subgroup with Lie algebra $\mathfrak{k}$.* □

This theorem has one flaw: There may be several Lie algebras with Cartan decomposition which determine the same symmetric space. In case of euclidean space $S = \mathbb{R}^n$, the semidirect product of the translation group together with any closed subgroup K of the orthogonal group is a subgroup G of the euclidean group $E(n)$ such that $S = G/K$; note that $\mathfrak{g} = \mathfrak{k} \oplus \mathbb{R}^n$ is a Cartan decomposition. However we will prove uniqueness of the Cartan decompositon if S is simply connected without flat factor (cf. Section 6).

If we wish uniqueness in general, we have to pass from Cartan decompositions to Lie triple systems. Recall that a *Lie triple system* is a euclidean vector space with a triple product $(x, y, z) \mapsto R(x, y)z$ which is antisymmetric in x, y and satisfies Bianchi's identity, such that all $R(x, y)$ are skew adjoint derivations of R. If we have a Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then $\mathfrak{p}$ with the triple product $R(x, y)z := [z, [x, y]]$ is a Lie triple system: Bianchi's identity is the Jacobi identity, and $R(x, y) = -ad([x, y])$ is a skew adjoint derivation of R since $[x, y] \in \mathfrak{k}$; recall that $\mathrm{Ad}(K)$ acts by automorphisms on the Lie algebra $\mathfrak{g}$ preserving $\mathfrak{p}$, so it acts by orthogonal automorphisms on $(\mathfrak{p}, R)$, and hence $\mathrm{ad}(\mathfrak{k})$ acts by skew adjoint derivations (cf. (L) at the end of Section 2).

Vice versa, if $(\mathfrak{p}, R)$ is a Lie triple system, we will recover a Lie algebra with Cartan decomposition by a construction due to E.Cartan: Let K be the connected component of the group of orthogonal automorphisms of $(\mathfrak{p}, R)$ and $\mathfrak{k}$ its Lie algebra which contains all $R(x, y)$. Let $\mathfrak{g} := \mathfrak{k} \oplus \mathfrak{p}$ as a vector space and define a Lie bracket on $\mathfrak{g}$ as follows. On $\mathfrak{k}$, the Lie bracket is already defined. The bracket $[\mathfrak{k}, \mathfrak{p}]$ is the action of $\mathfrak{k}$ on $\mathfrak{p}$, and for $x, y \in \mathfrak{p}$ we put $[x, y] := -R(x, y) \in \mathfrak{k}$. It is easy to check that $(\mathfrak{g}, [\ ,\ ])$ is a Lie algebra and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition. Recall that $\mathfrak{p}$ is already equipped with a K-invariant scalar product, hence the corresponding simply connected symmetric space is uniquely determined.

On the other hand, the Lie triple system $\mathfrak{p}$ is also uniquely determined by a symmetric space S: We may consider $\mathfrak{p}$ as a subspace of the full Lie algebra of Killing fields on S on which the conjugation with the symmetry s_p acts as $-I$; compare the construction before Theorem 4.2. Thus $\mathfrak{p}$ lies in the (-1)-eigenspace of this conjugation which is the space of infinitesimal transvections at p, and by reasons of dimension the two spaces agree. Thus we have seen:

Theorem 4.3 *Simply connected symmetric spaces and Lie triple systems are in one-to-one correspondence.* □

The practical achievement of this chapter is the following: If $\mathfrak{g}$ is a matrix Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, we can compute the curvature tensor of the corresponding symmetric space using Theorem 3, since the Lie triple $\mathfrak{p}$ is uniquely determined and the Lie brackets in $\mathfrak{g}$ can be computed in terms of matrices.

Example. Consider the complex projective space (cf. Example 6, Section 1)

$$S = \mathbb{C}P^n = G_1(\mathbb{C}^{n+1}) = U(n+1)/(U(1) \times U(n))$$

The Lie algebra of $G = U(n+1)$ is the space $\mathfrak{g}$ of antihermitean complex $(n+1) \times (n+1)$-matrices; the Cartan involution on $\mathfrak{g}$ is the conjugation with the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix},$$

hence $\mathfrak{p}$ is the set of matrices

$$X = \begin{pmatrix} 0 & -x^* \\ x & 0 \end{pmatrix}$$

for any $x \in \mathbb{C}^n$. For any $X, Y, Z \in \mathfrak{p}$ with corresponding $x, y, z \in \mathbb{C}^n$ we obtain $r(x, y)z \in \mathbb{C}^n$ corresponding to $R(X, Y)Z = [Z, [X, Y]] \in \mathfrak{p}$ (cf. Thm.3) by computing the matrix commutators:

$$r(x, y)z = z(y^*x - x^*y) + x(y^*z) - y(x^*z) \in \mathbb{C}^n$$

(recall that $y^*x \in \mathbb{C}$ is the hermitean scalar product between x and y in $\mathbb{C}^n$). In particular, for $y = z$ we get

$$r(x, y)y = y(y^*x - 2x^*y) + x(y^*y).$$

Let y be a unit vector, $y^*y = 1$, and x perpendicular to y with respect to the real scalar product $\langle x, y \rangle = \mathrm{Re}\ x^*y$. Then $y^*x = -x^*y = i\langle x, iy \rangle$ and hence

$$r(x, y)y = x + 3\langle x, iy \rangle iy.$$

Thus the eigenvalues of $r(.,y)y$ on $y^\perp$ are 1 on $\{y, iy\}^\perp$ and 4 on $\mathbb{R}iy$. Hence the sectional curvature of $\mathbb{C}P^n$ varies between 1 and 4 where the value 1 is taken on any real plane spanned by x, y with $x \perp y, iy$ and 4 on any complex plane spanned by y, iy.

5. Locally Symmetric Spaces

A (not necessarily complete) Riemannian manifold M is called *locally symmetric* if its curvature tensor is parallel, i.e. $\nabla R = 0$.

Theorem 5. *M is locally symmetric if and only if there exists a symmetric space S such that M is locally isometric to S.*

Proof Let M be locally isometric to a symmetric space S, i.e. around any $p \in M$ there is a geodesic ball $B = B_\epsilon(p)$ which is isometric to an ϵ-ball in S. Hence there is an isometry $s_p : B \to B$ fixing p with $(ds_p)_p = -I$. Let

$$w = (\nabla_{v_1} R)(v_2, v_3)v_4$$

for $v_1, ..., v_4 \in T_pM$. Applying $(ds_p)_p$ on both sides of this equation, w is changed to $-w$, and v_i to $-v_i$, hence the left hand side changes sign while the right hand side stays the same (4 minus signs); note that ∇R is preserved by $(ds_p)_p$. Thus $\nabla R = 0$.

Vice versa, assume $\nabla R = 0$ on M, i.e.

$$\nabla_W(R(X,Y)Z) = R(\nabla_W X, Y)Z + R(X, \nabla_W Y)Z + R(X,Y)\nabla_W Z$$

for all tangent vector fields X, Y, Z, W. Differentiating another time we see that this equation holds also with ∇_W replaced by $R(V, W)$ for arbitrary tangent vector fields V, W. In other words, $R(V, W)$ is a derivation of R (considered as a triple product on T_pM) and hence (T_pM, R) is a Lie triple system. By Theorem 4.2 there is a corresponding symmetric space S with curvature tensor $\hat{R}$ such that $(T_xS, \hat{R})$ and (T_pM, R) are orthogonally isomorphic Lie triples, for any $x \in S$ and $p \in M$. We identify $\mathfrak{p} := T_xS$ and T_pM by such an isomorphism. Let $\epsilon > 0$ such that $\hat{e} := \exp_x$ and $e := \exp_p$ are diffeomorphisms on $B_\epsilon(0)$. We claim that the diffeomorphism $\phi = \hat{e} \circ e^{-1} : B_\epsilon(p) \to B_\epsilon(x)$ is an isometry. We have to show that $|de_v(u)| = |d\hat{e}_v(u)|$ for any $v \in \mathfrak{p}$ and $u \in T_v\mathfrak{p}$.

We may write $w := de_v(u) = \frac{d}{ds}e(v + su)|_{s=0}$, hence w is the value $J(1)$ of the Jacobi field $J(t) = \frac{d}{ds}e(t(v + su))|_{s=0}$ along the geodesic $\gamma_v(t) = e(tv)$ in M. Then J satisfies the Jacobi equation $J'' + R(J, \gamma_v')\gamma_v' = 0$ with $J(0) = 0$ and $J'(0) = u$. Let $e_1, ..., e_n$ be an orthonormal basis of $\mathfrak{p} = T_pM$ and extend it to a parallel basis $E_1(t), ..., E_n(t)$ along γ_v. Then $J = \sum_i j_i E_i$ with $j_i = \langle J, E_i \rangle$. From $\nabla_{\gamma_v'} R = 0$ we see that $R(E_i, \gamma_v')\gamma_v'$ is parallel along γ_v, hence it can be written as $\sum_j r_{ji} E_j$ with constant coefficients $r_{ji} = \langle R(E_i, \gamma_v')\gamma_v', E_j \rangle$. Thus the vector valued function $j = (j_1, ..., j_n)^T$ satisfies the linear differential equation $j'' + rj = 0$ with $j(0) = 0$, $j'(0) = u$ where r is the constant matrix (r_{ji}) and $u = \sum u_i e_i$ is considered as the vector $(u_1, ..., u_n)^T$. Likewise, $\hat{w} := d\hat{e}_v(u)$ is the value $\hat{J}(1)$ of some Jacobi field $\hat{J}$ on S. But by the first part of this proof, the curvature tensor $\hat{R}$ of S is also parallel. Using the same orthonormal basis of $\mathfrak{p} = T_xS$, the Jacobi field $\hat{J}(t)$ is given by a vector valued function $\hat{j}(t)$ satisfying precisely the same initial value problem. By the unicity theorem for ODEs we have $\hat{j}(t) = j(t)$ and thus $|\hat{J}(t)| = |J(t)|$ for all t. In particular we get $|\hat{w}| = |w|$ which finishes the proof. $\square$

6. Compact, Noncompact, Euclidean Type; Duality

There is a prominent symmetric bilinear form on each Lie algebra $\mathfrak{g}$, the *Killing form*, defined as follows. Any representation $\rho : G \to GL(V)$ determines a symmetric bilinear form b_ρ on $\mathfrak{g}$, namely $b_\rho(x, y) = \operatorname{trace} \rho_*(x)\rho_*(y)$. This is $\mathrm{Ad}(G)$-invariant since $\rho_*(\mathrm{Ad}(g)x) = \rho(g)\rho_*(x)\rho(g)^{-1}$. The *Killing form* B is this bilinear form for the adjoint representation, $B = b_{\mathrm{Ad}}$, i.e.

$$B(x, y) = \operatorname{trace} \operatorname{ad}(x) \operatorname{ad}(y).$$

Since B is $\mathrm{Ad}(G)$-invariant, all $\operatorname{ad}(x)$ are skew symmetric with respect to B, i.e. $B([x, y], z) = -B(y, [x, z])$ or $B(z, [x, y]) = B([z, x], y)$.

If $K \subset G$ is a compact subgroup, then there exists an $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{g}$ (start with any scalar product and take its average over K). Thus for any $x \in \mathfrak{k}$, the endomorphism $\operatorname{ad}(x) = \frac{d}{dt} \mathrm{Ad}(\exp tx)|_{t=0}$ is skew symmetric with respect to this scalar product, so its square $\operatorname{ad}(x)^2$ is symmetric with nonpositive eigenvalues. Hence $B(x, x) = \operatorname{trace} \operatorname{ad}(x)^2 < 0$ unless $\operatorname{ad}(x) = 0$ which means that x is in the center of $\mathfrak{g}$. Thus $-B|_{\mathfrak{k}}$ is a positive definite scalar product unless $\mathfrak{k}$ intersects the center of $\mathfrak{g}$. But this latter case is impossible if G acts *locally effectively* on G/K (i.e. no $g \neq e$ close to e acts as identity): If $k \in K \cap Z(G)$ (where $Z(G)$ denotes the center of G), then $k(gK) = gkK = gK$ for any $g \in G$, and so k acts trivially on G/K.

Now let $S = G/K$ be an isotropy irreducible symmetric space, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Since $\mathfrak{p}$ may be identified with the tangent space of S at the base point $p = eK$, the Riemannian metric on S corresponds to a K-invariant scalar product $\langle\ ,\ \rangle$ on $\mathfrak{p}$. But there is yet another K-invariant symmetric bilinear form on $\mathfrak{p}$, namely the restriction of the Killing form B. Since K acts irreducibly on $\mathfrak{p}$, the two symmetric bilinear forms differ only by a factor λ (since any eigenspace of B is K-invariant), i.e. $B = \lambda \cdot \langle\ ,\ \rangle$. The sign of λ will determine the *type* of the symmetric space.

Theorem 6 *The sectional curvature of an isotropy irreducible symmetric space S is zero for $\lambda = 0$, and for $\lambda \neq 0$ it is*

$$\langle R(x, y)y, x\rangle = \lambda^{-1} B([x, y], [x, y])$$

where $x, y \in \mathfrak{p}$ are orthonormal.

Proof Recall from Section 3 that $R(x, y)z = [z, [x, y]]$ for all $x, y \in \mathfrak{p} = T_pS$. Thus

$$\lambda\langle R(x, y)y, x\rangle = B(x, R(x, y)y) = B(x, [y, [x, y]]) = B([x, y], [x, y]).$$

If $\lambda \neq 0$, this proves the claim. If $\lambda = 0$, the above equation implies $[x, y] = 0$: Recall that $[x, y] \in \mathfrak{k}$ for $x, y \in \mathfrak{p}$, thus $B([x, y], [x, y]) \leq 0$ and $= 0$ only if $[x, y] = 0$. Thus the full curvature tensor vanishes in this case. □

If $\lambda = 0$, then S is flat (curvature zero) and is called of *euclidean type*. It is easy to see that such a space is the Riemannian product of some euclidean space $\mathbb{R}^k$ with a flat $(n-k)$-torus $(0 \leq k \leq n)$.

If $\lambda < 0$, then S has nonnegative sectional curvature and is called of *compact type*. In fact, since S is isotropy irreducibe, it has constant Ricci curvature which must be positive (if it were zero, the sectional curvature had to vanish). Thus S and all its covering spaces must be compact by Myers' theorem; in particular, the fundamental group of S is finite.

If $\lambda > 0$, the curvature is nonpositive; such a symmetric space S is called of *noncompact type*. By the theorem of Hadamard and Cartan, it is diffeomorphic to $\mathbb{R}^n$ if it is simply connected. In fact, S *is* simply connected. Otherwise, there would be a closed geodesic γ: A curve minimizing the length in a nontrivial homotopy class of loops starting and ending at the base point $p \in S$ is a geodesic loop, but any geodesic in S is the orbit of a one-parameter subgroup (cf. Section 2), hence a geodesic loop is already a closed geodesic. Any Killing field X gives a Jacobi field $\xi = X \circ \gamma$ along the geodesic γ, i.e. $\xi'' + R(\xi, \gamma')\gamma' = 0$. Hence from the curvature condition we get $\langle \xi'', \xi \rangle \geq 0$ and therefore $\langle \xi, \xi \rangle'' = 2(\langle \xi'', \xi \rangle + \langle \xi', \xi' \rangle) \geq 0$. Thus $\langle \xi, \xi \rangle$ is a convex function, but it is also periodic, being the restriction of the function $\langle X, X \rangle$ to a closed geodesic. So $\langle \xi, \xi \rangle = \mathit{const}$ and $\langle R(\xi, \gamma')\gamma', \xi \rangle = 0$. Therefore, any 2-plane containing γ' has curvature zero. In particular, the Ricci curvature in direction γ' vanishes. Since the Ricci curvature is constant, it vanishes everywhere, and since the sectional curvature is nonpositive, this implies that the curvature is zero.

Symmetric spaces of compact and of noncompact type are related by the so called *duality*: From any Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ we get another Lie algebra $\mathfrak{g}'$ which is just $\mathfrak{g}$ as a vector space, with the following Lie bracket $[\ ,\ ]'$: for any $x, y \in \mathfrak{p}$ and $a, b \in \mathfrak{k}$ we have

$$[a, b]' = [a, b], \quad [a, x]' = [a, x], \quad [x, y]' = -[x, y].$$

It is easy to check that this is again a Lie algebra with the same Cartan decomposition, but the sign of the curvature tensor of the corresponding symmetric space S is reversed.*) Starting from a symmetric space of compact type we obtain one of noncompact type, called the *dual symmetric space*, and vice versa. E.g. the spheres and projective spaces are dual to the corresponding hyperbolic spaces, and the positive definite symmetric matrices $GL(n, \mathbb{R})/O(n)$ are dual to the space $U(n)/O(n)$ of real structures in $\mathbb{C}^n$ (cf. examples in Section 1). The dual of a compact Lie group $S = G = (G \times G)/G$ (cf. Example 5) is G^c/G where G^c corresponds to the complexified Lie algebra $\mathfrak{g}^c = \mathfrak{g} \otimes \mathbb{C}$.

7. The Isometry Group

Recall that an *ideal* $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is a linear subspace with $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$, and the *center* of $\mathfrak{g}$ is the largest subspace $\mathfrak{c}$ with $[\mathfrak{g}, \mathfrak{c}] = 0$. A trivial Lie algebra (all of whose Lie brackets vanish) is called *abelian*. A nonabelian Lie algebra without proper nonzero ideal is called *simple*. A Lie algebra $\mathfrak{g}$ is called *semisimple* if its Killing form in nondegenerate. The reason for the last notion is the following Lemma:

Lemma. *If a Lie algebra $\mathfrak{g}$ is semisimple, it has no center and any ideal $\mathfrak{a} \subset \mathfrak{g}$ has a complementary ideal $\mathfrak{b}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, or equivalently, $\mathfrak{g}$ splits uniquely as a direct sum of simple ideals.*

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*) In fact, we can change $[x, y]$ to $\alpha[x, y]$ for *any* $\alpha \in \mathbb{R}$. If $\alpha \neq 0$, we can assume $\alpha = \pm 1$; otherwise we just scale the metric of S by $|\alpha|$. For $\alpha = 0$, we always obtain euclidean space.

Proof Let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal. We let $\mathfrak{b} := \mathfrak{a}^\perp = \{x \in \mathfrak{g};\ B(x, \mathfrak{a}) = 0\}$ be the orthogonal complement of $\mathfrak{a}$ with respect to B. This is also an ideal, since for all $b \in \mathfrak{a}^\perp$, $x \in \mathfrak{g}$ and $a \in \mathfrak{a}$ we have $B([b, x], a) = B(b, [x, a]) = 0$ since $[x, a] \in \mathfrak{a}$. It remains to show that $\mathfrak{c} := \mathfrak{a} \cap \mathfrak{a}^\perp = 0$. In fact, $\mathfrak{a} \cap \mathfrak{a}^\perp$ is an ideal, and it is abelian since $B([a, b], x) = B(a, [b, x]) = 0$ for all $a, b \in \mathfrak{a} \cap \mathfrak{a}^\perp$ and $x \in \mathfrak{g}$ which implies $[a, b] = 0$. But there is no abelian ideal $\mathfrak{c}$ if B is nondegenerate (in particular, there is no center): The linear map $\mathrm{ad}(x) ad(c)$ takes values in $\mathfrak{c}$ for any $x \in \mathfrak{g}$ and $c \in \mathfrak{c}$, so $B(x, c)$ is the trace of $\mathrm{ad}(x) ad(c)$ taken over the subspace $\mathfrak{c}$, but this is zero since $\mathrm{ad}(x)\,\mathrm{ad}(c)c' = 0$ for any $c' \in \mathfrak{c}$. Thus $B(x, c) = 0$ for all $x \in \mathfrak{g}$, hence $c = 0$.

By further decomposing the ideals $\mathfrak{a}$ and $\mathfrak{b} = \mathfrak{a}^\perp$, we eventually arrive at a decomposition $\mathfrak{g} = \mathfrak{a}_1 \oplus ... \oplus \mathfrak{a}_p$ where all $\mathfrak{a}_i$ are simple ideals. This decomposition is unique (up to permutation of the factors) since the factors $\mathfrak{a}_i$ are irreducible and inequivalent under the action of $\mathrm{ad}(\mathfrak{g})$ and $\mathrm{Ad}(G)$ where G is the corresponding Lie group.*) $\square$

Theorem 7.1 *If $S = G/K$ is a symmetric space of compact or noncompact type, then $\mathfrak{g}$ is semisimple. If moreover S is strongly isotropy irreducible, i.e. the isotropy representation of the connected component K^o is irreducible, then either $\mathfrak{g}$ is simple or S is a simple compact Lie group with biinvariant metric (cf. Example 5, Section 1).*

Proof Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. By Section 6, B is (positive or negative) definite on $\mathfrak{p}$ and $\mathfrak{k}$. Further, $B(\mathfrak{p}, \mathfrak{k}) = 0$ since $\mathfrak{p}$ and $\mathfrak{k}$ are the eigenspaces of the automorphism σ_* of $\mathfrak{g}$, and clearly any automorphism leaves B invariant. Thus B is nondegenerate and hence $\mathfrak{g}$ semisimple.

Thus by the previous Lemma, $\mathfrak{g}$ splits uniquely as a direct sum of simple ideals $\mathfrak{g} = \mathfrak{a}_1 \oplus ... \oplus \mathfrak{a}_p$ which are permuted by the Cartan involution σ_*, i.e. $\sigma_*(\mathfrak{a}_i) = \mathfrak{a}_{s(i)}$. Putting $\mathfrak{b}_i = \mathfrak{a}_i + \mathfrak{a}_{s(i)}$, we get a σ_*-invariant splitting $\mathfrak{g} = \mathfrak{b}_1 \oplus ... \oplus \mathfrak{b}_q$. Thus $\mathfrak{p}$ and $\mathfrak{k}$ split accordingly. By strong irreducibility we must have $q = 1$, hence either $\mathfrak{g}$ is simple or $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}$ where $\mathfrak{a}$ is simple and $\sigma_*(a, b) = (b, a)$ for any $a, b \in \mathfrak{a}$. In the latter case $\mathfrak{k}$ the fixed point set $\mathfrak{k}$ of σ_* is the diagonal $\{(a, a);\ a \in \mathfrak{a}\}$, and thus S is the compact Lie group K with a biinvariant metric. $\square$

Theorem 7.2 *Let S be a symmetric space of compact or noncompact type and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Lie algebra of all Killing fields. Then $\mathfrak{p}$ generates $\mathfrak{g}$ as a Lie algebra. Moreover the Lie algebras of the holonomy and the isotropy group agree, and its representation on the tangent space determines S uniquely up to coverings and duality.*

Proof Let $a \in \mathfrak{k}$ be B-perpendicular to the linear span of $\{[x, y];\ x, y \in \mathfrak{p}\}$ where B is the Killing form. Then we have $0 = B(a, [x, y]) = B([a, x], y)$ for all $x, y \in \mathfrak{p}$ which shows $[a, x] = 0$ for all $x \in \mathfrak{p}$. Thus $\exp ta$ acts as identity on $\mathfrak{p}$ which shows $a = 0$ since the action is effective. Hence $\mathfrak{g}$ is generated by $\mathfrak{p}$ as a Lie algebra.

We have already seen that the isotropy group (acting faithfully on the tangent space $\mathfrak{p} = T_pS$ by the adjoint action) contains the holonomy group (Section 2). But the Lie algebra of the holonomy group contains the endomorphisms $\mathrm{ad}([x, y])|_{\mathfrak{p}} = -R(x, y)$ of $\mathfrak{p}$ for all $x, y \in \mathfrak{p}$, and these endomorphisms generate $\mathfrak{k}$ (acting on $\mathfrak{p}$).

It remains to show that the action of $\mathfrak{k}$ on $\mathfrak{p}$ determines the Lie bracket on $\mathfrak{p}$ up to a factor. This is done similarly: The Lie bracket $[x, y] \in \mathfrak{k}$ for $x, y \in \mathfrak{p}$ is determined by the scalar products $B(a, [x, y])$ for all $a \in \mathfrak{k}$, but $B(a, [x, y]) = B([a, x], y) = \lambda\langle [a, x], y\rangle$ (cf. Section 6) is determined by the action of $\mathfrak{k}$ on $\mathfrak{p}$ and the factor λ. $\square$

*) This is true even if $\mathfrak{a}_i \cong \mathfrak{a}_j$ for some $i \neq j$ since $\mathrm{ad}(\mathfrak{a}_j)$ acts trivially on $\mathfrak{a}_i$ but not so on $\mathfrak{a}_j$.

8. Lie Subtriples and Totally Geodesic Subspaces

To any symmetric space $S = G/K$ we have assigned a Lie triple $(\mathfrak{p}, R)$ with $R(x,y)z = [z,[x,y]]$ (cf Theorem 4). A *Lie subtriple* is a linear subspace $\mathfrak{p}' \subset \mathfrak{p}$ which is invariant under the triple product R. Considered as a Lie triple in its own right, a Lie subtriple corresponds again to a symmetric space. On the other hand, a complete totally geodesic (immersed) submanifold S' of S is clearly invariant under all symmetries s_p for $p \in S'$, hence it is also symmetric. We shall see that these two objects are closely related:

Theorem 8. *Let $S = G/K$ be a symmetric space and $S' \subset S$ a complete (possibly immersed) submanifold passing through $p = eK$. Then the following statements are equivalent:*
(a) $S' \subset S$ is totally geodesic.
(b) $S' \subset S$ is a symmetric subspace, i.e. $s_q(S') = S'$ for all $q \in S'$.
(c) $S' = \exp_p \mathfrak{p}'$ where $\mathfrak{p}' \subset \mathfrak{p} = T_pS$ is a Lie subtriple.

Proof "(a) $\Rightarrow$ (c)": Any totally geodesic submanifold through p must be of the form $\exp_p(\mathfrak{p}')$ for some linear subspace $\mathfrak{p}' \subset \mathfrak{p} = T_pS$ which is invariant under the curvature tensor, hence a Lie subtriple.

"(c) $\Rightarrow$ (b)": Let $\mathfrak{p}' \subset \mathfrak{p}$ be a Lie subtriple and $S' = \exp_p \mathfrak{p}'$. Note that $(\exp \mathfrak{p}')p = \exp_p \mathfrak{p}'$ (where exp denotes the Lie group exponential while $\exp_p$ is the Riemannian exponential map), since for any $X \in \mathfrak{p}$ the geodesic $\gamma_X(t) = \exp_p tX$ is the orbit through p of the one-parameter group $g(t) = \exp tX$ (see Section 2). From the Lie triple property of $\mathfrak{p}'$ we get that $\mathfrak{g}' := \mathfrak{p}' + [\mathfrak{p}', \mathfrak{p}'] \subset \mathfrak{g}$ is a Lie subalgebra. Let $G' \subset G$ be the corresponding connected Lie subgroup. Since $[\mathfrak{p}', \mathfrak{p}'] \subset \mathfrak{k}$, we have $\mathfrak{g}'p = \mathfrak{p}'p$ and thus the orbit $G'p$ has tangent space $T_pS' = \mathfrak{p}'$. Moreover we have $S' \subset G'p$ with $T_pS' = \mathfrak{p}' = T_p(G'p)$ whence $S' = G'p$ by completeness. By definition, S' is invariant under s_p and then by homogeneity under s_q for any $q \in S'$, see Theorem 1.

"(b) $\Rightarrow$ (a)": If $s_q(S') = S'$ for any $q \in S'$, then the second fundamental form α must vanish since $-\alpha(v,w) = ds_q(\alpha(v,w)) = \alpha(ds_q.v, ds_q.w) = \alpha(v,w)$ for any $v, w \in T_qS'$. $\square$

Corollary 1. *Let $S' \subset S$ be complete and totally geodesic. Then all transvections of S' (considered as a symmetric space of its own right) extend uniquely to transvections of S.* $\square$

Corollary 2. *The flat complete totally geodesic subspaces of S are precisely $S' = \exp \mathfrak{p}'$ where $\mathfrak{p}' \subset \mathfrak{p} \subset \mathfrak{g}$ is an abelian subalgebra.**)

Proof Recall that $[\mathfrak{p}', \mathfrak{p}'] \subset \mathfrak{g}'$ (notation as in the proof of the above theorem). So $S' = G'/K'$ is a symmetric space of euclidean type if and only if $[\mathfrak{p}', \mathfrak{p}'] = 0$ (cf. Sect.6).

9. Isotropy Representation and Rank

The easiest symmetric space is of course euclidean space. What can be said about totally geodesic immersions of euclidean space into an arbitrary symmetric space $S =$

*) Remember that $\mathfrak{p}$ is not a Lie algebra. We mean a subalgebra of $\mathfrak{g}$ which is contained in $\mathfrak{p}$. Such a subalgebra $\mathfrak{a}$ is automatically abelian since $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a} \cap [\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{p} \cap \mathfrak{k} = 0$.

G/K? Such (immersed) submanifolds are called *flats*. We will consider only *maximal* flats which are not contained in any larger flat. By the results of the previous chapter, maximal flats through the base point point p are precisely $F = \exp_p(\mathfrak{a})$ for any maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$, or equivalently, F is the orbit $A.p$ of the subgroup $A = \exp \mathfrak{a} \subset G$ corresponding to the Lie algebra $\mathfrak{a}$.

We show first that F is a closed submanifold of S. It is sufficient to prove that A is a closed subgroup of G. In fact, A is maximal among the connected abelian subgroups of G with Lie algebra contained in $\mathfrak{p}$, and these properties also hold for its closure $\bar{A}$: Clearly $\bar{A}$ is again abelian and connected; we must show only that its Lie algebra is also in $\mathfrak{p}$. But this property can be re-interpreted: $\mathfrak{a}$ is a subspace of $\mathfrak{p}$ iff the automorphism $\sigma = Ad(s_p)$ acts on it as $-I$. This means that $s_p g s_p^{-1} = g^{-1}$ for any $g \in A$, in other words, the conjugation with the symmetry s_p is the inversion on A. This property of A clearly passes over to the closure $\bar{A}$. Hence $\bar{A} = A$ by maximality of A. Since K is compact, the projection of $A \subset G$ to $S = G/K$ is closed, too. If S is compact, F is also compact (and homogeneous and flat), hence it is a flat torus.

In order to investigate the maximal flats it is sufficient to consider the maximal abelian subalgebras of $\mathfrak{p}$. Any two of them are conjugate by the isotropy group:

Theorem 9. *Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra. Then $\mathfrak{a}$ intersects any K-orbit on $\mathfrak{p}$, and each time it intersects perpendicularly.*

Proof We show first that $\mathfrak{a}$ intersects K-orbits perpendicularly. Let $x \in \mathfrak{a}$ and consider the orbit $O = \mathrm{Ad}(K)x \subset \mathfrak{p}$ as a submanifold of the euclidean vector space $\mathfrak{p}$. Its tangent space is $T_xO = T_x(\mathrm{Ad}(K)x) = \mathrm{ad}(\mathfrak{k})x$. Hence its normal space $\nu_x O$ contains all vectors $y \in \mathfrak{p}$ with $0 = \langle [\mathfrak{k}, x], y\rangle = \langle \mathfrak{k}, [x, y]\rangle$ which means $[x, y] = 0$. Thus the normal vectors of $O \subset \mathfrak{p}$ at x are those which commute with x; in particular, $\mathfrak{a} \subset \nu_x O$.

In the Lemma below we show that in fact $\mathfrak{a} = \nu_x O$ for almost every (so called *regular*) $x \in \mathfrak{a}$. Pick such a regular vector $x \in \mathfrak{a}$. Then $\nu_{\mathrm{Ad}(k)x} O = \mathrm{Ad}(k)\mathfrak{a}$ for all $k \in K$. Let $\mathfrak{a}' \subset \mathfrak{p}$ be another maximal abelian subalgebra and let $y \in \mathfrak{a}'$ be regular in $\mathfrak{a}'$. Since O is a compact submanifold of $\mathfrak{p}$, there exists a point $\mathrm{Ad}(k)x \in O$ (for some $k \in K$) which has smallest distance to y. Thus the line segment from y to $\mathrm{Ad}(k)x$ is perpendicular to the manifold O, i.e. $y - Ad(k)x$ lies in $\nu_{Ad(k)x} O = Ad(k)\mathfrak{a}$. Hence y lies also in the maximal abelian subalgebra $\mathrm{Ad}(k)\mathfrak{a}$. By the Lemma below we have $\mathfrak{a}' = Ad(k)\mathfrak{a}$ which finishes the proof. $\square$

Lemma *There is a finite number of hyperplanes of $\mathfrak{a}$ such that any $x \in \mathfrak{a}$ which lies outside these hyperplanes is regular, i.e. for any $y \in \mathfrak{p}$ we have $[x, y] = 0$ only if $y \in \mathfrak{a}$. A regular vector lies in precisely one maximal abelian subalgebra.*

Proof We may assume that S is of compact or noncompact type. We have to consider the endomorphisms $\mathrm{ad}(x) \in \mathrm{End}(\mathfrak{g})$ for all $x \in \mathfrak{a}$. These are diagonalizable over $\mathbb{C}$: From the Killing form B on $\mathfrak{g}$ we get a positive definite scalar product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ which is $-B$ on $\mathfrak{k}$ and $\pm B$ on $\mathfrak{p}$ (cf. Section 6), and for all $x \in \mathfrak{a}$, $y \in \mathfrak{p}$ and $k \in \mathfrak{k}$ we have

$$\langle \mathrm{ad}(x)y, k\rangle = -B(\mathrm{ad}(x)y, k) = B(y, \mathrm{ad}(x)k) = \pm\langle y, \mathrm{ad}(x)k\rangle$$

(recall that $\mathrm{ad}(x)y \in \mathfrak{k}$ and $\mathrm{ad}(x)k \in \mathfrak{p}$). Hence $\mathrm{ad}(x)$ is self adjoint in the noncompact case (with real eigenvalues) and skew adjoint in the compact case (with imaginary eigenvalues). For sake of simplicity, let us assume that S is of noncompact type (which is no restriction of generality since the maximal abelian subalgebras of $\mathfrak{p}$ and the isotropy representation remain the same under duality).

Since $\mathfrak{a}$ is abelian, the endomorphisms $\mathrm{ad}(x)$ for $x \in \mathfrak{a}$ commute with each other (Jacobi identity). Thus we get a common eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha \tag{1}$$

where $\Delta \subset \mathfrak{a}^*$ is a finite set of nonzero real linear forms on $\mathfrak{a}$ (called *roots*), and

$$\mathfrak{g}_\alpha = \{z \in \mathfrak{g};\ \forall_{x \in \mathfrak{a}}\ \ \mathrm{ad}(x)z = \alpha(x)z\}.$$

(A nonzero linear form $\alpha \in \mathfrak{a}^*$ is called a *root* if this space $\mathfrak{g}_\alpha$ is nonzero.) Now recall that $\mathrm{ad}(x)$ for $x \in \mathfrak{a} \subset \mathfrak{p}$ interchanges the factors of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Splitting each eigenvector $z_\alpha \in \mathfrak{g}_\alpha$ as $z_\alpha = x_\alpha + y_\alpha$ with $x_\alpha \in \mathfrak{p}$ and $y_\alpha \in \mathfrak{k}$, we get from $[x, z_\alpha] = \alpha(x)z_\alpha$:

$$[x, x_\alpha] = \alpha(x)y_\alpha, \quad [x, y_\alpha] = \alpha(x)x_\alpha. \tag{2}$$

(For compact type, the eigenvalues of $\mathrm{ad}(x)$ are $i\alpha(x)$ and we obtain a minus sign in one of these equations.) Putting $\bar{z}_\alpha = \sigma_*(z_\alpha) = -x_\alpha + y_\alpha$, we get further from (2) that $[x, \bar{z}_\alpha] = -\alpha(x)\bar{z}_\alpha$; hence $-\alpha$ is also a root, and we have a splitting

$$\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} = \mathfrak{k}_\alpha + \mathfrak{p}_\alpha \tag{3}$$

where $\mathfrak{k}_\alpha \subset \mathfrak{k}$ contains the y_α and $\mathfrak{p}_\alpha \subset \mathfrak{p}$ the x_α component of z_α. Now from (1) and (3) we obtain a decomposition

$$\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Delta} \mathfrak{p}_\alpha.$$

(In fact we may replace Δ by a subset Δ_+ of half cardinality containing just one of any two roots $\pm\alpha$, see below.)

Now we consider the hyperplanes $\alpha^\perp = \ker \alpha \subset \mathfrak{a}$. If $x \in \mathfrak{a} \setminus \bigcup_\alpha \alpha^\perp$, then x does not commute with any $x' \in \mathfrak{p} \setminus \mathfrak{a}$ since we may split $x' = x_0 + \sum x_\alpha$ with $x_0 \in \mathfrak{a}$ and $x_\alpha \in \mathfrak{p}_\alpha$, and $[x, x'] = \sum \alpha(x)y_\alpha \neq 0$. Thus these vectors are regular. In particular, any such x lies in no other maximal abelian subalgebra of $\mathfrak{p}$. □

The hyperplanes $\alpha^\perp \subset \mathfrak{a}$ are called *root hyperplanes*. The connected components of $\mathfrak{a} \setminus \bigcup_\alpha \alpha^\perp$ are called *Weyl chambers*. On a Weyl chamber C, a root α does not change sign, i.e. it takes either positive or negative values on all of C. In the sequel, we will call $x \in \mathfrak{a}$ *regular* if it does not lie in any root hyperplane, i.e. it is in the union of the (open) Weyl chambers.

Fixing a Weyl chamber C, we can choose Δ_+ to be the set of roots which are positive on C (called "*positive roots*"); then apparently $\Delta = \Delta_+ \cup -\Delta_+$.

As a consequence of Theorem 9, all maximal abelian subalgebras of $\mathfrak{p}$ and hence all maximal flats in S have the same dimension which is called the *rank of the symmetric space S*. We have also seen that the rank is the codimension of the principal orbits (which we called O in the proof above) of the isotropy representation. If the rank is one, these orbits are the spheres in $\mathfrak{p}$ and the space is called *two point homogeneous* since any two tangent vectors of the same length or any two pairs of points of the same distance can be mapped upon each other by an isometry. These spaces are the spheres and the projective and hyperbolic spaces. In the higher rank case, the maximal abelian subalgebra $\mathfrak{a}$ contains a system of representatives for the classes of tangent vectors which are equivalent under isometries.

The orbits of the isotropy representation of a symmetric space S are interesting spaces in its own right and they have been extensively studied (e.g. cf. [KN], [BR], [PT]). Among them are the extrinsic symmetric spaces which were mentioned in Section 1; these are precisely the orbits $\mathrm{Ad}(K)x$ where $\mathrm{ad}(x)^3 = \mu \cdot \mathrm{ad}(x)$ for some nonzero $\mu \in \mathbb{R}$ (cf. [F], [EH]). Representations all of whose orbits meet a certain subspace perpendicularly are called *polar*; we have seen that isotropy representations of symmetric spaces have this property. J.Dadok [D] has proved the converse statement: For any polar representation there exists an isotropy representation of a symmetric space ("*s-representation*") having the same orbits. Dadok's classification of polar representations can be viewed as an alternative approach to symmetric spaces and their classification.

10. The Weyl Group

Let $S = G/K$ be a symmetric space of compact or noncompact type, G connected, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. Let $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subalgebra. We have seen in the last chapter that $\mathfrak{a}$ intersects every orbit of the isotropy group K (acting on $\mathfrak{p}$ by Ad). However, each orbit $\mathrm{Ad}(K)x$ has several intersection points with $\mathfrak{a}$. In fact, consider the compact subgroup

$$M = \{k \in K;\ \mathrm{Ad}(k)\mathfrak{a} = \mathfrak{a}\} \subset K.$$

Obviously we have $\mathrm{Ad}(M)x \subset \mathrm{Ad}(K)x \cap \mathfrak{a}$ for any $x \in \mathfrak{a}$. Equality holds if $x \in \mathfrak{a}$ is regular:

$$\mathrm{Ad}(K)x \cap \mathfrak{a} = \mathrm{Ad}(M)x.$$

To prove this, let $k \in K$ with $\mathrm{Ad}(k)x \in \mathfrak{a}$. Since x and $\mathrm{Ad}(k)x$ are regular, the normal spaces of $O = \mathrm{Ad}(K)x$ at x and $\mathrm{Ad}(k)x$ both are equal to $\mathfrak{a}$. Thus $\mathrm{Ad}(k)$ preserves $\mathfrak{a}$ since it maps $\nu_x O$ onto $\nu_{\mathrm{Ad}(k)x} O$, and hence $k \in M$.

Since M leaves $\mathfrak{a}$ invariant, it acts on $\mathfrak{a}$, and the kernel of this action is apparently the subgroup

$$M_0 = \{k \in K; \mathrm{Ad}(k)x = x\ \forall x \in \mathfrak{a}\}$$

The groups M and M_0 have the same Lie algebra

$$\mathfrak{m} = \{y \in \mathfrak{k};\ [y, \mathfrak{a}] = 0\}.$$

This is because $[y, \mathfrak{a}] \subset \mathfrak{a}$ already implies $[y, \mathfrak{a}] = 0$ since $B([y, \mathfrak{a}], x) = B(y, [\mathfrak{a}, x]) = 0$ for all $x \in \mathfrak{a}$. Therefore the quotient group $W = M/M_0$ is discrete and hence finite (recall that M was compact); it is called the *Weyl group* of S with respect to $\mathfrak{a}$. Since the root system depends only on the choice of $\mathfrak{a}$, the action of M or W on $\mathfrak{a}$ permutes the root hyperplanes and maps Weyl chambers to Weyl chambers (recall that a Weyl chamber was a connected component of $\mathfrak{a} \setminus \bigcup_\alpha \alpha^\perp$). The following theorem determines the size of W.

Theorem 10. *The Weyl group W acting on $\mathfrak{a}$ is generated by the reflections at the root hyperplanes $\alpha^\perp = \ker \alpha$ for all $\alpha \in \Delta$, and it acts simply transitively on the set of Weyl chambers in $\mathfrak{a}$.*

Proof (cf. [H], pp. 283-289) We show first that the reflection at the root hyperplane $\alpha^\perp$ is contained in W for any $\alpha \in \Delta$. Put $\Delta_0 = \Delta \cup \{0\}$ and let $\mathfrak{p}_0 = \mathfrak{a}$. The subspace

$$\mathfrak{p}^\alpha = \sum_{\beta \in \Delta_0 \cap \mathbb{R}\alpha} \mathfrak{p}_\beta$$

is a Lie subtriple of $\mathfrak{p}$. This follows from the Jacobi identity which implies $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ for any $\alpha, \beta \in \Delta_0$: In fact, for $z_\alpha \in \mathfrak{g}_\alpha$ and $z_\beta \in \mathfrak{g}_\beta$ and all $x \in \mathfrak{a}$ we have

$$[x, [z_\alpha, z_\beta]] = [[x, z_\alpha], z_\beta] + [z_\alpha, [x, z_\beta]] = (\alpha(x) + \beta(x))[z_\alpha, z_\beta].$$

This Lie subtriple $\mathfrak{p}^\alpha$ essentially corresponds to a rank-one symmetric space. In fact, the hyperplane $\alpha^\perp = \ker \alpha \subset \mathfrak{a}$ can be split off from $\mathfrak{a} \subset \mathfrak{p}^\alpha$ since $\mathfrak{g}_\beta$ commutes with $\ker \alpha$ for any $\beta \in \Delta_0 \cap \mathbb{R}\alpha$, and further

$$\langle \alpha^\perp, [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}] \rangle = \lambda^{-1} B(\alpha^\perp, [\mathfrak{g}_\beta, \mathfrak{g}_{-\beta}]) = \lambda^{-1} B([\alpha^\perp, \mathfrak{g}_\beta], \mathfrak{g}_{-\beta}) = 0.$$

Let $\mathfrak{p}''$ be the orthogonal complement of $\alpha^\perp$ in $\mathfrak{p}^\alpha$. Then the totally geodesic subspace S' corresponding to the Lie triple $\mathfrak{p}^\alpha$ splits locally as $S'' \times \mathbb{R}^{k-1}$ where the second factor has tangent space $\alpha^\perp$, and S'' is a rank-one symmetric space with Lie triple $\mathfrak{p}''$. Thus the connected component K'' of the isotropy group of S'' acts transitively on the unit sphere in $\mathfrak{p}''$; in particular there is some $k \in K''$ which maps the dual vector $\alpha^* \in \mathfrak{a}$ (with $\alpha(\alpha^*) = 1$ and $\alpha^* \perp \alpha^\perp$) onto $-\alpha^*$. By Theorem 7.2, the isometry group of S'' is generated by transvections and thus all elements K'' extend to isometries of S' (acting trivially on the $\mathbb{R}^{k-1}$-factor) and further to isometries of S. In this way, K'' is embedded into K^o. The particular element $k \in K''$ lies in M since $\mathrm{Ad}(k)$ preserves $\mathfrak{a}$, and since $\mathrm{Ad}(k)$ fixes $\alpha^\perp$ and reflects α^*, it is the reflection at the root hyperplane $\alpha^\perp$.

Since the group generated by the reflections at all root hyperplanes acts transitively on the set of Weyl chambers it remains to show that any $w \in W$ which leaves a Weyl chamber C invariant acts trivially. Pick such a w and let r be its order. For an arbitrary $x' \in C$ consider the "w-average" $x := x' + wx' + ... + w^{r-1}x'$ which is also in C since C is a convex cone. Apparently we have $wx = x$, and since x is regular, the claim follows from the subsequent lemma. □

Lemma. *For any $k \in K$ we have: If $\mathrm{Ad}(k)$ fixes some regular element $x \in \mathfrak{a}$ then $\mathrm{Ad}(k)$ fixes every element of $\mathfrak{a}$.*

Proof By duality it is no restriction of generality to assume that S is of compact type, i.e. $S = G/K$ with G compact and semisimple. Since $\alpha(x) \neq 0$ for any $\alpha \in \Delta$, we see from the decomposition

$$\mathfrak{g} = \mathfrak{k}_0 + \sum \mathfrak{k}_\alpha + \mathfrak{a} + \sum \mathfrak{p}_\alpha$$

that the centralizer Lie algebra of x (containing the elements of $\mathfrak{g}$ which commute with x) is precisely $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}$, and this is contained in the centralizer Lie algebra of *any* element of $\mathfrak{a}$. Thus we only have to show that $k \in \exp\, \mathfrak{g}_0$. More generally, we will show that for any $x \in \mathfrak{g}$ the centralizer subgroup $C(x) = \{g \in G;\ \mathrm{Ad}(g)x = x\}$ is connected.

In fact, let T_x be the closure of $\exp \mathbb{R}x$. This is a closed connected abelian subgroup ("*torus*") in G. Let $c \in C(x)$ be arbitrary. Then c commutes with T_x, hence c and T_x generate a closed abelian subgroup $A \subset G$. Its connected component is some torus A^o

(containing T_x). Thus there is some $a \in A^o$ such that A^o is the closure of $\langle a \rangle = a^{\mathbb{Z}}$. By compactness, A has only finitely many connected components. Hence there is a power p such that $c^p \in A^o$. and so A is the closure of $\langle ac \rangle$ since $\overline{\langle ac \rangle}$ contains T_x and c. Because G is compact and connected, there is some $y \in \mathfrak{g}$ with $ac = \exp y$. Now the closure of $\exp \mathbb{R}y$ is a torus T_y in G containing c and T_x. Thus $T_y \subset C(x)$ which shows that c lies in the connected component of $C(x)$. □

Now we have gained a rather complete picture of the isotropy representation of symmetric spaces of compact or noncompact type: Any maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ is an orthogonal section of all orbits. Further, $\mathfrak{a}$ is a union of the closures of open polyhedral (even simplicial) cones called Weyl chambers. The orbits through the interior of these cones are precisely the *regular* ones, those of maximal dimension (exceptional orbits don't exist); they intersect each Weyl chamber exactly once. In a rank-one symmetric space, any two unit tangent vectors can be mapped onto each other by an isometry. For a symmetric space S of higher rank, the Weyl chambers (in any maximal abelian subalgebra at any point of S) take the place of the unit tangent vectors: Any two of them can be mapped onto each other by an isometry.

The roots α corresponding to the root hyperplanes $\alpha^{\perp}$ which bound a fixed Weyl chamber C are called *fundamental roots*; since the Weyl group permutes the Weyl chambers, all roots are conjugate to the fundamental roots under the Weyl group action. The possible fundamental root systems and the corresponding symmetric spaces are classified; cf. tables in [H], pp. 476, 518, 532ff.

References

[B] A. Besse: *Einstein manifolds*, Springer 1985

[BR] F.E. Burstall, J.H. Rawnsley: *Twistor Theory for Riemannian Symmetric Spaces*, Springer L.N. in Math. **1424**, 1990

[CE] J. Cheeger, D.G. Ebin: *Comparison Theorems in Riemannian Geometry*, North Holland 1975

[D] J. Dadok: *Polar coordinates induced by actions of compact Lie groups*, Transactions A.M.S. **288**, 125 - 137 (1985)

[E] P. Eberlein: *Manifolds of Nonpositive Curvature*, Section 2. Chicago University Press 1996

[EH] J.-H. Eschenburg, E. Heintze: *Extrinsic symmetric spaces and orbits of s-representations*, manuscripta math. **88**, 517 - 524 (1995)

[F] D. Ferus, *Symmetric submanifolds of Euclidean space*, Math Ann. **247** (1980), 81 - 93

[H] S. Helgason: *Differential Geometry, Lie Groups. and Symmetric Spaces*, Academic Press 1978

[Hi] U. Hirzebruch: *Über Jordan-Algebren und kompakte Riemannsche symmetrische Räume vom Rang 1*, Math. Z. **90**, 339 - 354 (1965)

[KN] S. Kobayashi, T. Nagano: *On filtered Lie algebras and geometric structures I*, J. Math. Mech. 13, 875 - 907 (1964)

[KNo] S. Kobayashi, K. Nomizu: *Foundations of Differential Geometry, I, II*, Wiley 1973

[L] O. Loos: *Symmetric Spaces I, II*, Benjamin 1969

[PT] R. S. Palais, C.-L. Terng: *Critical Point Theory and Submanifold Geometry*, Springer L.N. in Math. **1353**, 1988